

Discrete Time Financial Model

Chapter 4: Discrete Time Financial Model

Agenda

- 1 Discrete Time Financial Model for Derivative Pricing
- 2 Principle of No-Arbitrage Pricing
- 3 State price vector and risk neutral probabilities
- 4 Physical Probability v.s. Risk Neutral Probability
- 5 Binomial Model
- 6 Multi-period Binomial models
- 7 Pricing in Practice
- 8 Summary

Discrete time financial model

Consider a financial market formed by n basic components called **financial assets/instruments**, denoted by $S_1, \dots, S_n$.

Definition 1 (Discrete time financial model)

- ① At **initial** time 0: n **assets** with values $S^0 = (S_1^0, \dots, S_n^0)'$.
- ② At **maturity** time T : m **scenarios**.
 - In scenario i ($1 \leq i \leq m$), asset S_j ($1 \leq j \leq n$) has value S_j^i .
 - The information of the assets at T can be represented by

$$S = \begin{pmatrix} S_1^1 & S_2^1 & \cdots & S_n^1 \\ S_1^2 & S_2^2 & \cdots & S_n^2 \\ S_1^3 & S_2^3 & \cdots & S_n^3 \\ \vdots & \vdots & & \vdots \\ S_1^m & S_2^m & \cdots & S_n^m \end{pmatrix}.$$

We assume that $S_j^i \geq 0$ for all i and j , since an asset has no liability.

The Pricing Problem

$$\text{Market: } \left\{ \mathbf{S}^0 = \begin{pmatrix} S_1^0 \\ \vdots \\ S_n^0 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} S_1^1 & S_2^1 & \dots & S_n^1 \\ S_1^2 & S_2^2 & \dots & S_n^2 \\ S_1^3 & S_2^3 & \dots & S_n^3 \\ \vdots & \vdots & \dots & \vdots \\ S_1^m & S_2^m & \dots & S_n^m \end{pmatrix} \right\}, \quad n\text{-assets, } m\text{-scenarios}$$

Definition 2 (The Pricing Problem)

Consider a market $\{\mathbf{S}^0, \mathbf{S}\}$, in which a financial instrument F takes value

- F^i at the i -th scenario at maturity T , where $i = 1, \dots, m$.

The pricing problem refers to determining the price F^0 , which is the value of F at time 0.

- The term “pricing” in CAPM $\mu_a = r + \beta_a(\mu_M - r)$ refers to determining the **expected return** of an asset a .

Terminologies: Portfolio; Payoff; Complete

$$\text{Market: } \left\{ \mathbf{S}^0 = \begin{pmatrix} S_1^0 \\ \vdots \\ S_n^0 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} S_1^1 & S_2^1 & \dots & S_n^1 \\ S_1^2 & S_2^2 & \dots & S_n^2 \\ S_1^3 & S_2^3 & \dots & S_n^3 \\ \vdots & \vdots & \dots & \vdots \\ S_1^m & S_2^m & \dots & S_n^m \end{pmatrix} \right\}, \quad n\text{-assets, } m\text{-scenarios}$$

Definition 3

- **Portfolio:** $\sum_{j=1}^n w_j S_j$
 - A collection of assets with **weight vector** $\mathbf{w} = (w_1, \dots, w_n)'$
 - holds w_j units of asset S_j
- **Payoff Vector / Payoff:** $\mathbf{F} \in \mathbb{R}^m$
 - A vector that represents the values of a portfolio/financial instrument at the m different scenarios **at maturity**
- **Complete**
 - A market is **complete** if every payoff vector is attainable by a portfolio
 - i.e., For every $\mathbf{F} \in \mathbb{R}^m$, there exists $\mathbf{w} \in \mathbb{R}^n$ such that $\mathbf{F} = \mathbf{S}\mathbf{w}$.

Terminologies: Arbitrage

Definition 4 (Arbitrage)

An **arbitrage** is a portfolio w that has

- non-positive initial value ($w'S^0$, a scalar)
- non-negative value at maturity (Sw , an m -dimensional vector)
- excluding the case $w'S^0 = 0$ and $Sw = 0$

Mathematically, an arbitrage is a portfolio w that satisfies

$$\underbrace{w'S^0}_{\text{Price}} \leq 0 \text{ and } \underbrace{Sw}_{\text{Payoff}} > 0 \quad \text{or} \quad \underbrace{w'S^0}_{\text{Price}} < 0 \text{ and } \underbrace{Sw}_{\text{Payoff}} \geq 0,$$

- $Sw \geq 0$ means all entries of Sw are non-negative.
- $Sw > 0$ means all entries of Sw are non-negative with at least one element being positive.
- $Sw \gg 0$ means all entries of Sw are positive.

Terminologies: No Arbitrage

Arbitrage: $\underbrace{\mathbf{w}'\mathbf{S}^0}_{\text{Price}} \leq 0$ and $\underbrace{\mathbf{S}\mathbf{w}}_{\text{Payoff}} > 0$ or $\underbrace{\mathbf{w}'\mathbf{S}^0}_{\text{Price}} < 0$ and $\underbrace{\mathbf{S}\mathbf{w}}_{\text{Payoff}} \geq 0$

Basic Logic:

- If A and B are statements, and $\sim A$ denotes the negation of A
- then the following two statements are logically equivalent:

$$\textcircled{1} \quad \sim (A \text{ and } B) \quad \textcircled{2} \quad B \Rightarrow \sim A.$$

Definition 5 (No Arbitrage)

No arbitrage opportunity exists if the negation of Arbitrage's definition holds.

i.e., for all $\mathbf{w} \in \mathbb{R}^n$,

$$\underbrace{\mathbf{S}\mathbf{w}}_{\text{Payoff}} > 0 \Rightarrow \underbrace{\mathbf{w}'\mathbf{S}^0}_{\text{Price}} > 0 \quad \text{and} \quad \underbrace{\mathbf{S}\mathbf{w}}_{\text{Payoff}} \geq 0 \Rightarrow \underbrace{\mathbf{w}'\mathbf{S}^0}_{\text{Price}} \geq 0$$

Example: Completeness

Complete market: all payoff attainable: $\mathbf{F} = \mathbf{S}\mathbf{w}$.

Example 6 (Completeness)

Consider markets A , B and C having $\mathbf{S}_A^0 = \mathbf{S}_B^0 = \mathbf{S}_C^0 = (1, 1)'$, and

$$\mathbf{S}_A = \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ 0 & 3 \end{pmatrix}, \mathbf{S}_B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \mathbf{S}_C = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}.$$

- Market A: Not complete
 - Payoffs $\mathbf{F} = (a, b, c)'$ with $a \neq b$ cannot be replicated.
- Market B: Complete
 - Any payoff $\mathbf{F}$ can be replicated by $\mathbf{w} = \mathbf{S}_B^{-1}\mathbf{F}$
 - $\mathbf{S}_B\mathbf{w} = \mathbf{S}_B\mathbf{S}_B^{-1}\mathbf{F} = \mathbf{F}$
- Market C: Not complete
 - Payoffs $\mathbf{F} = (a, b)'$ with $2a \neq b$ cannot be replicated.

Example: Arbitrage

Arbitrage: $\underbrace{\mathbf{w}'\mathbf{S}^0}_{\text{Price}} \leq 0$ and $\underbrace{\mathbf{S}\mathbf{w}}_{\text{Payoff}} > \mathbf{0}$ or $\underbrace{\mathbf{w}'\mathbf{S}^0}_{\text{Price}} < 0$ and $\underbrace{\mathbf{S}\mathbf{w}}_{\text{Payoff}} \geq \mathbf{0}$

No Arbitrage: $\underbrace{\mathbf{S}\mathbf{w}}_{\text{Payoff}} > \mathbf{0} \Rightarrow \underbrace{\mathbf{w}'\mathbf{S}^0}_{\text{Price}} > 0$ and $\underbrace{\mathbf{S}\mathbf{w}}_{\text{Payoff}} \geq \mathbf{0} \Rightarrow \underbrace{\mathbf{w}'\mathbf{S}^0}_{\text{Price}} \geq 0$

Example 7 (Arbitrage)

$$\mathbf{S}_A^0 = \mathbf{S}_B^0 = \mathbf{S}_C^0 = (1, 1)', \quad \mathbf{S}_A = \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \mathbf{S}_B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{S}_C = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}.$$

- Market A: No arbitrage:

- To have payoff $\mathbf{S}_A \mathbf{w} = (2w_1, 2w_1, 3w_2)' > \mathbf{0}$, need $(w_1, w_2) > \mathbf{0}$
- Thus, initial price $\mathbf{w}'\mathbf{S}^0 = w_1 + w_2 > 0$.

- Market B: will be discussed later using [State Price Vector](#).

- Market C: Asset 1 vs Asset 2:

- same initial price
- Asset 1 has a better payoff in all scenarios

⇒ Arbitrage Portfolio: $\mathbf{w} = (1, -1)'$

- holding Asset 1 and shorting Asset 2 (buy low sell high)
- Price $\mathbf{w}'\mathbf{S}^0 = 1 - 1 = 0$ but payoff $\mathbf{S}\mathbf{w} = (1, 2)' > \mathbf{0}$

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Principle of No-Arbitrage Pricing

In finance, the **Principle of No-Arbitrage Pricing** is the key to pricing financial derivatives.

Definition 8 (Principle of No-Arbitrage Pricing)

To price a financial instrument F , we find a portfolio $(w_1, \dots, w_n)'$ such that

$$\sum_{j=1}^n w_j S_j^i = F^i, \quad i = 1, \dots, m,$$

i.e., the portfolio's and F 's payoffs match at every scenario at the maturity T .

• Explanation:

- Now F and $\sum_{j=1}^n w_j S_j$ have the same payoff structure at T
- If their initial prices differ, then “buy low sell high” makes an arbitrage
- Therefore, the price is $F^0 = \mathbf{w}'\mathbf{S}^0$, the initial value of the replicating portfolio.

Principle of No-Arbitrage Pricing

The procedure for No-Arbitrage Pricing:

Step 1. (Replication). To price F , find a *replicating portfolio* $\mathbf{w} = (w_1, w_2, \dots, w_n)'$ s.t.

$$\mathbf{F} \triangleq \begin{pmatrix} F^1 \\ F^2 \\ F^3 \\ \vdots \\ F^m \end{pmatrix} = \begin{pmatrix} S_1^1 & S_2^1 & \dots & S_n^1 \\ S_1^2 & S_2^2 & \dots & S_n^2 \\ S_1^3 & S_2^3 & \dots & S_n^3 \\ \vdots & \vdots & & \vdots \\ S_1^m & S_2^m & \dots & S_n^m \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{pmatrix} \triangleq \mathbf{S}\mathbf{w}.$$

i.e., holding $(S_1, S_2, \dots, S_n)'$ with weight $(w_1, \dots, w_n)'$ replicates the payoff of F in every scenario.

Step 2: (Pricing). The price of the instrument, F^0 , is given by

$$F^0 = \mathbf{w}'\mathbf{S}^0 = \sum_{j=1}^n w_j S_j^0.$$

Assumptions

$$\mathbf{S} = \begin{pmatrix} S_1^1 & S_2^1 & \dots & S_n^1 \\ S_1^2 & S_2^2 & \dots & S_n^2 \\ S_1^3 & S_2^3 & \dots & S_n^3 \\ \vdots & \vdots & & \vdots \\ S_1^m & S_2^m & \dots & S_n^m \end{pmatrix}$$

Assumption 9 (Linearly Independent Columns of S)

S has linearly independent columns.

- Otherwise, at least one column is a linear combination of other columns. (some assets can be constructed by other assets)

Assumption 10

Number of scenarios is the same as number of assets, i.e., $m = n$.

- Ensure that the inverse $\mathbf{S}^{-1}$ of $\mathbf{S}$ exists.

Replicating Portfolio

Theorem 11 (Replicating Portfolio)

If S has linearly independent columns and $m = n$, then for any given payoff vector F , the weight of replicating portfolio is given by

$$\mathbf{w} = \mathbf{S}^{-1}\mathbf{F}.$$

As a replicating portfolio exists for any F , the market is complete.

Proof.

- Every square matrix ($m = n$) with linearly independent columns is invertible.
- With $\mathbf{w} = \mathbf{S}^{-1}\mathbf{F}$, we have $\mathbf{S}\mathbf{w} = \mathbf{S}\mathbf{S}^{-1}\mathbf{F} = \mathbf{F}$.



Pricing

Using the weight $\mathbf{w} = \mathbf{S}^{-1}\mathbf{F}$, we obtain a simple pricing formula:

Corollary 12 (Pricing)

For any payoff $\mathbf{F}$ and asset structure $\{\mathbf{S}^0, \mathbf{S}\}$, the price is

$$F^0 \triangleq (\mathbf{S}^0)' \mathbf{S}^{-1} \mathbf{F}.$$

Proof.

Given the replicating portfolio $\mathbf{w}$, the price is $F^0 = \mathbf{w}' \mathbf{S}^0$.
Since F^0 is a scalar, we can write

$$F^0 = (F^0)' = (\mathbf{S}^0)' \mathbf{w} = (\mathbf{S}^0)' \mathbf{S}^{-1} \mathbf{F}.$$



Example

Corollary: The price of a payoff $\mathbf{F}$ is $F^0 \triangleq (\mathbf{S}^0)' \mathbf{S}^{-1} \mathbf{F}$.

Example 13

$$\mathbf{S}_A^0 = \mathbf{S}_B^0 = \mathbf{S}_C^0 = (1, 1)', \quad \mathbf{S}_A = \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \mathbf{S}_B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{S}_C = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}.$$

- Only Market B satisfies the assumptions of $m = n$ and linear independence.
- Given a payoff $\mathbf{F} = (3, 6)'$ in Market B , the replicating portfolio is

$$\mathbf{w} = \mathbf{S}_B^{-1} \mathbf{F} = (0, 3)'.$$

- The price of F is given by $F^0 = (\mathbf{S}^0)' \mathbf{w} = (1, 1) \cdot (0, 3)' = 3$.

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State price vector

- Recall the pricing formula $F^0 \triangleq (\mathbf{S}^0)' \mathbf{S}^{-1} \mathbf{F}$.

- Define the **state price vector** (an m -dimensional vector) as

$$\boldsymbol{\psi}' = (\mathbf{S}^0)' \mathbf{S}^{-1}.$$

- The pricing formula can be rewritten as

$$F^0 = \boldsymbol{\psi}' \mathbf{F} = \sum_{j=1}^m \psi_j F^j. \quad (1)$$

- The structure in (1) is similar to taking expectation where
 - $\psi_1, \dots, \psi_m$ are “probabilities” (not exactly, as $\sum_{j=1} \psi_j \neq 1$ in general)
 - $F^1, \dots, F^m$ are possible values in the sample space.

Properties of State Price Vector

State Price Vector: $\psi' = (\mathbf{S}^0)' \mathbf{S}^{-1}$

Pricing: $F^0 = \psi' \mathbf{F} = \sum_{j=1}^m \psi_j F^j$

Properties of State Price Vector (SPV):

① SPV lists the **price** of \$ 1 at each **state**:

- $\mathbf{F} = (F^1, \dots, F^m)' = (0, \dots, 1, \dots, 0)'$ (k -th entry) is \$ 1 at State k
- Price $F^0 = \sum_{j=1}^m \psi_j F^j = \psi_k$

② SPV recovers initial prices from maturity prices: $(\mathbf{S}^0)' = \psi' \mathbf{S}$

Proof: $(\mathbf{S}^0)' = (\mathbf{S}^0)' \mathbf{S}^{-1} \mathbf{S} = \psi' \mathbf{S}$

③ Sum of entries of SPV is discount factor: $\bar{\psi} := \sum_{j=1}^m \psi_j = \frac{1}{1+r}$

Proof: $\mathbf{F} = (1, \dots, 1)'$ is a \$1 risk free asset $\Rightarrow$ The price is $F^0 = \frac{1}{1+r}$

But $F^0 = \sum_{j=1}^m \psi_j F^j = \sum_{j=1}^m \psi_j$

Risk Neutral Probability

State Price Vector: $\psi' = (\mathbf{S}^0)' \mathbf{S}^{-1}$

- ① $F^0 = \psi' \mathbf{F} = \sum_{j=1}^m \psi_j F^j$
- ② ψ_k = price for \$ 1 at State k
- ③ $(\mathbf{S}^0)' = \psi' \mathbf{S}$
- ④ $\bar{\psi} := \sum_{j=1}^m \psi_j = \frac{1}{1+r}$ is the **discount factor**

- Since $\sum_{j=1}^m \psi_j \neq 1$ in general, ψ is not a probability distribution.
- To obtain a probability distribution, we rescale ψ to define the

Risk Neutral (RN) Probability Vector : $\tilde{\psi} \triangleq \psi / \bar{\psi}$.

- ① **Price:** $F^0 = \psi' \mathbf{F} = \bar{\psi} \tilde{\psi}' \mathbf{F} = \bar{\psi} \left(\sum_{j=1}^m \tilde{\psi}_j F^j \right) =: \bar{\psi} \mathbf{E}_{\tilde{\psi}}(\mathbf{F})$
(discounted expectation)
- ② $\tilde{\psi}_k$ is the **risk neutral probability** on State k for pricing
- ③ $(\mathbf{S}^0)' = \bar{\psi} \tilde{\psi}' \mathbf{S} \xrightarrow{i-th} S_i^0 = \bar{\psi} \sum_{j=1}^m \tilde{\psi}_j S_i^j \implies \sum_{j=1}^m \tilde{\psi}_j \frac{S_i^j}{S_i^0} = \frac{1}{\bar{\psi}}, \forall i$
 - Expected “gain” of each stock equals the same value $\frac{1}{\bar{\psi}}$ (risk “neutral”)
- ④ $\sum_{j=1}^m \tilde{\psi}_j = (\sum_{j=1}^m \psi_j) / \bar{\psi} = \bar{\psi} / \bar{\psi} = 1$ (probability indeed!)

State Price Vector and No arbitrage

Proposition 14

There is no arbitrage opportunity in the market if and only if the state price vector satisfies $\psi \gg 0$. (every entry is positive)

$$\text{No Arbitrage:} \quad \underbrace{\mathbf{F} = \mathbf{S}\mathbf{w}}_{\text{Payoff}} > \mathbf{0} \Rightarrow \underbrace{F^0 = \mathbf{w}'\mathbf{S}^0}_{\text{Price}} > 0$$

Proof.

- Suppose that $\psi \gg 0$.
 - If the payoff $\mathbf{F} = \mathbf{S}\mathbf{w} > \mathbf{0}$, then the price is $F^0 = \psi'\mathbf{F} > 0$.
 $\Rightarrow$ No arbitrage condition is satisfied.
- Suppose that there is no arbitrage opportunity
 - Consider a payoff $\mathbf{F} = (0, \dots, 0, 1, 0, \dots, 0)'$. (k -th entry = 1)
 - No arbitrage $\Rightarrow$ price of $\mathbf{F}$ is positive $\Rightarrow F^0 = \psi'\mathbf{F} = \psi_k > 0$
 - Repeat the same argument for $k = 1, \dots, m$, we have $\psi \gg 0$.

State Price Vector and Risk Neutral Prob.: Example

State Price Vector: $\psi' = (\mathbf{S}^0)' \mathbf{S}^{-1}$

Risk Neutral Prob. Vector: $\tilde{\psi} = \psi / \bar{\psi}$

- $F^0 = \psi' \mathbf{F} = \bar{\psi} \tilde{\psi}' \mathbf{F}$, where $\bar{\psi} = \sum_{j=1}^m \psi_j$ is discount factor
- No arbitrage $\Leftrightarrow \psi \gg 0$.

Example 15

$$\mathbf{S}_B^0 = (1, 1)', \quad \mathbf{S}_B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

- **State Price Vector:** $\psi' = (\mathbf{S}_B^0)' \mathbf{S}_B^{-1} = (1/3, 1/3)$.
 - Since $\psi \gg 0$, there is no arbitrage in the market.
- **Discount factor:** $\bar{\psi} = 1/3 + 1/3 = 2/3$
- **Risk Neutral Prob:** $\tilde{\psi} = \psi / \bar{\psi} = (1/3, 1/3)' / (2/3) = (1/2, 1/2)'$
- Price of a derivative with payoff $\mathbf{F} = (3, 6)'$ is $F^0 = \psi' \mathbf{F} = \bar{\psi} \tilde{\psi}' \mathbf{F} = 3$.
 - **SPV:** \$ 3 at State 1 $(3 \times \frac{1}{3})$ + \$ 6 at State 2 $(6 \times \frac{1}{3}) = 3$
 - **RNP:** Discounting $(\frac{2}{3} \times) \tilde{\psi}$ -expected payoff $(3 \times \frac{1}{2} + 6 \times \frac{1}{2}) = 3$

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Physical Probability v.s. Risk Neutral Probability

State Price Vector: $\psi' = (\mathbf{S}^0)' \mathbf{S}^{-1}$

Risk Neutral Prob. Vector: $\tilde{\psi} = \psi / \bar{\psi}$

• $F^0 = \psi' \mathbf{F} = \bar{\psi} \tilde{\psi}' \mathbf{F}$, where $\bar{\psi} = \sum_{j=1}^m \psi_j$ is discount factor

- **Physical probabilities** only measures **chance of occurrence** of states
- **RNP** measures **overall utility** of states derived from the market structure $(\mathbf{S}^0, \mathbf{S})$
- If a financial derivative is replicable by $\mathbf{S}$, pricing without RNP leads to arbitrage

Example 16 (Physical Probability v.s. Risk Neutral Probability)

Consider the market: $\mathbf{S}^0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{S} = \begin{pmatrix} 1.2 & 10 \\ 1.2 & 1 \end{pmatrix}$, $\mathbf{F} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$.

$P(\text{Scenario 1 occurs}) = 0.9$; $P(\text{Scenario 2 occurs}) = 0.1$; $r = 0.2$.

- Payoff $\mathbf{F}$ corresponds to a **call option** of Asset 2 at strike price \$2.
- **SPV:** $\psi' = (\mathbf{S}^0)' \mathbf{S}^{-1} = (0.019, 0.815)$; **RNP:** $\tilde{\psi} = \psi / \bar{\psi} = (0.022, 0.978)'$
- **Physical Price:** $\bar{\psi} E_P(\mathbf{F}) = \$[8(0.9) + 0(0.1)] / 1.2 = \$6$
- **RN Price:** $\bar{\psi} E_{\tilde{\psi}}(\mathbf{F}) = \$[8(0.022) + 0(0.978)] / 1.2 = \$0.148$
- If call is \$ 6, then buy replicating portfolio & sell call $\Rightarrow$ \$ (6-0.148) arbitrage profit
- RNP $\tilde{\psi}$ infers from market structure that State 2 should weight more (see Asset 2)

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Definition 17

Binomial model is a discrete time financial model with $n = m = 2$.

- two assets: bond (B) & stock (S).
- two scenarios: upward (u) & downward (d) movements of S ($u > d$)

Specifically,

$$\mathbf{S}^0 = (1, s)' \quad \text{and} \quad \mathbf{S} = \begin{pmatrix} e^{rT} & su \\ e^{rT} & sd \end{pmatrix}.$$

- s : initial price of stock
- r : continuously compounded risk-free interest rate for the bond
- e^{rT} : accumulation factor.

Properties of Binomial Model

$$\text{Binomial Model: } \mathbf{S}^0 = (1, s)' \text{ and } \mathbf{S} = \begin{pmatrix} e^{rT} & su \\ e^{rT} & sd \end{pmatrix}.$$

Theorem 18 (Properties of Binomial Model)

a) *Replicating Portfolio* for payoff $\mathbf{F} = (F^1, F^2)' \triangleq (F_u, F_d)'$:

$$\mathbf{w} = \mathbf{S}^{-1} \mathbf{F} = \left(\frac{F_d u - F_u d}{e^{rT}(u - d)}, \frac{F_u - F_d}{s(u - d)} \right)',$$

b) *Risk Neutral Probabilities*. The state price vector, discount factor and risk neutral probability are uniquely given by

$$\psi = ((\mathbf{S}^0)' \mathbf{S}^{-1})' = \frac{1}{e^{rT}(u - d)} (e^{rT} - d, u - e^{rT})',$$

$$\bar{\psi} = \sum_{j=1}^2 \psi_j = e^{-rT}, \quad \tilde{\psi} = \psi / \bar{\psi} = \left(\frac{e^{rT} - d}{u - d}, \frac{u - e^{rT}}{u - d} \right)'.$$

c) *Pricing*: $F^0 = \psi' \mathbf{F} = \frac{F_u(e^{rT} - d) + F_d(u - e^{rT})}{e^{rT}(u - d)}.$

d) *Arbitrage* exists if $u > d > e^{rT}$ or $d < u < e^{rT}$.

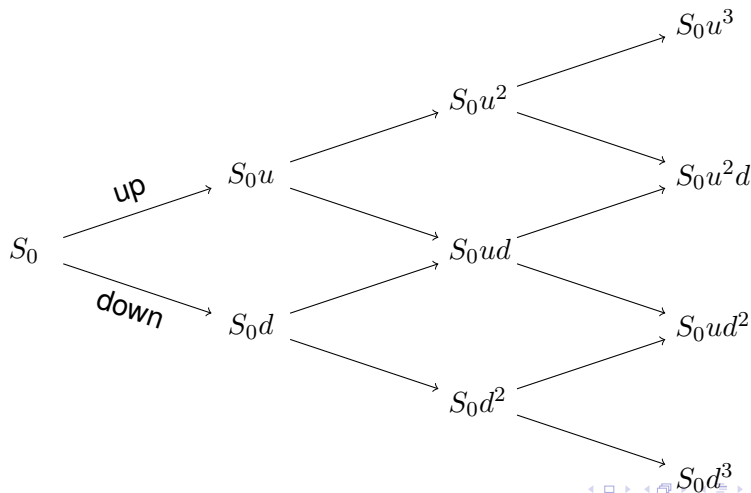
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Multi-period Binomial Model

Multi-period Binomial model/tree

- bring together copies of binomial models
- allows more possibilities at maturity T



Multi-period Binomial Model

Definition 19 (Multi-period Binomial Model)

n -period binomial model:

- market is observable at $0 = t_0 < t_1 < t_2 < \dots < t_n = T$, ($t_k = k\delta$).
- Each $[t_k, t_{k+1}]$: a binomial model with a bond and a stock.
- Time 0: $(B_0, S_0) = (1, s)$,

$$\text{Time } t_k : \quad B_k = e^{rt_k}, \quad S_k = \begin{cases} S_{k-1}u & \text{if market goes up} \\ S_{k-1}d & \text{if market goes down} \end{cases}$$

- can handle 2^n possible scenarios without specifying 2^n assets.
- some of the 2^n scenarios are the same (e.g., $udd = dud = ddu$).
- need to treat 2^n possible scenarios differently for path dependent instruments (e.g. up-and-out option).

State Price Vector at each branch

n-period binomial model: $(B_0, S_0) = (1, s)$

Time $t_k = k\delta$: $B_k = e^{rt_k}$, $S_k = \begin{cases} S_{k-1}u & \text{if market goes up} \\ S_{k-1}d & \text{if market goes down} \end{cases}$.

Theorem 20 (State Price Vector at each branch)

Suppose that at time t_k a branch of the tree takes value (e^{rt_k}, S_k) . The binomial model at this branch can be expressed in matrix form by

$$\mathbf{S}^0 = (e^{rt_k}, S_k)' \quad \text{and} \quad \mathbf{S} = \begin{pmatrix} e^{rt_{k+1}} & S_k u \\ e^{rt_{k+1}} & S_k d \end{pmatrix}.$$

Note that $t_{k+1} - t_k = \delta$. The state price vector is given by

$$\begin{aligned} \psi' &= (\mathbf{S}^0)' \mathbf{S}^{-1} = \frac{1}{e^{r\delta}(u-d)} \begin{pmatrix} e^{r\delta} - d, & u - e^{r\delta} \end{pmatrix} \\ &=: (\psi_u, \psi_d). \end{aligned}$$

By construction, the state price vector is common for all branches.

n -period binomial model: $(B_0, S_0) = (1, s)$

Time $t_k = k\delta$: $B_k = e^{rt_k}$, $S_k = \begin{cases} S_{k-1}u & \text{if market goes up} \\ S_{k-1}d & \text{if market goes down} \end{cases}$.

- To discuss the asset dynamic, we need the notion of **path**

Definition 21 (Path in an n -period binomial model)

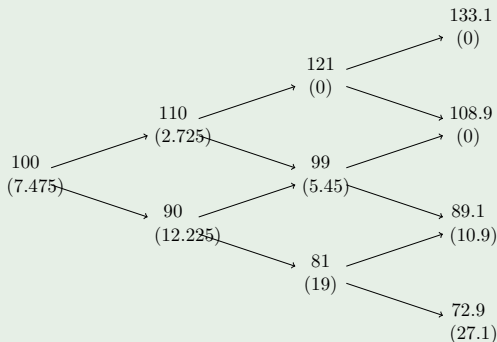
A **path** $\omega_t = (Z_1, \dots, Z_t)'$ up to time $t = 1, \dots, n$ is

- a t -dimensional vector specifying the evolution of the market up to time t
- for $i = 1, \dots, t$, $Z_i = \begin{cases} 1 & \text{if market goes up,} \\ -1 & \text{if market goes down.} \end{cases}$
- Once ω_n is given, we know ω_t for $t \leq n$
- There are 2^n possible paths, denoted as $\{w_n^j\}_{j=1, \dots, 2^n}$

Path: $\omega_t = (Z_1, \dots, Z_t)'$, $Z_i = \begin{cases} 1 & \text{if market goes up,} \\ -1 & \text{if market goes down.} \end{cases}$

Example 22 (Path)

Consider stock prices S_t
(derivative prices F_t)
in the 3-period model



- If the stock price goes up in the first two periods, then $\omega_2 = (1, 1)'$.
- $S_1(\omega_2) = 110$, $S_2(\omega_2) = 121$; $F_1(\omega_2) = 2.725$, $F_2(\omega_2) = 0$.
- $S_3(\omega_2)$ and $F_3(\omega_2)$ are unknown

Pricing in Multi-period Binomial model

$$\text{Path: } \omega_t = (Z_1, \dots, Z_t)', \quad Z_i = \begin{cases} 1 & \text{up; state price } \psi_u = \frac{e^{r\delta} - d}{e^{r\delta}(u-d)} \\ -1 & \text{down; state price } \psi_d = \frac{u - e^{r\delta}}{e^{r\delta}(u-d)} \end{cases}$$

Theorem 23 (Pricing in Multi-period Binomial model)

- Consider possible payoffs of F at time n : $F^j, j = 1, \dots, 2^n$
- All possible paths: $\omega_n^j = (Z_1^j, \dots, Z_n^j)', j = 1, \dots, 2^n$
- State Price Vector**: $\Psi = (\Psi^1, \dots, \Psi^{2^n})'$, where

$$\Psi^j = \prod_{t=1}^n \left(\psi_u \mathbb{I}_{\{Z_t^j=1\}} + \psi_d \mathbb{I}_{\{Z_t^j=-1\}} \right)$$

is the **state price** associated with the path ω_n^j .

- Discount factor**: $\bar{\Psi} \triangleq \sum_{j=1}^{2^n} \Psi^j = \sum_{k=0}^n C_k^n \psi_u^k \psi_d^{n-k} = (\psi_u + \psi_d)^n = e^{-rT}$
- Risk Neutral Probability Vector**: $\tilde{\Psi} = \bar{\Psi}^{-1} \Psi$
- Price of F at $t = 0$** : $F^0 = \sum_{j=1}^{2^n} \Psi^j F^j = \bar{\Psi} \sum_{j=1}^{2^n} \tilde{\Psi}^j F^j \triangleq \bar{\Psi} E_{\tilde{\Psi}}(F)$

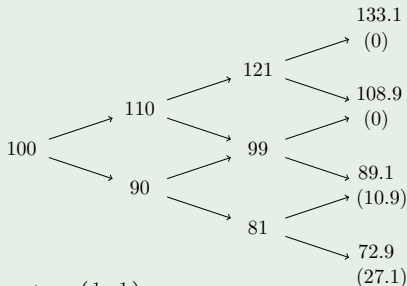
Pricing in Multi-period Binomial Model

Price of F at $t = 0$: $F^0 = \sum_{j=1}^{2^n} \Psi^j F^j$

• $\Psi^j = \prod_{t=1}^n \left(\psi_u \mathbb{I}_{\{Z_t^j=1\}} + \psi_d \mathbb{I}_{\{Z_t^j=-1\}} \right)$; • $(\psi_u, \psi_d) = e^{-r\delta} \left(\frac{e^{r\delta} - d}{(u-d)}, \frac{u - e^{r\delta}}{(u-d)} \right)$

Example 24 (Pricing in Multi-period Binomial Model)

Consider stock prices S_t
(derivative prices F_t)
in the 3-period model.
Let risk-free rate $r = 0$.
What is the price F^0 ?



- Note that $u = 1.1$, $d = 0.9$, $(\psi_u, \psi_d) = \left(\frac{1}{2}, \frac{1}{2}\right)$
- $\Psi^1 = 0.5^3$ (3 $\uparrow$); $\Psi^2 = \Psi^3 = \Psi^4 = 0.5 \times 0.5^2$ (2 $\uparrow$ 1 $\downarrow$);
 $\Psi^5 = \Psi^6 = \Psi^7 = 0.5^2 \times 0.5$ (1 $\uparrow$ 2 $\downarrow$); $\Psi^8 = 0.5^3$ (3 $\downarrow$)
- Price:
 $F^0 = \Psi^1 \times 0 + (\Psi^2 + \Psi^3 + \Psi^4) \times 0 + (\Psi^5 + \Psi^6 + \Psi^7) \times 10.9 + \Psi^8 \times 27.1 = 7.475$

Agenda

- 1 Discrete Time Financial Model for Derivative Pricing
- 2 Principle of No-Arbitrage Pricing
- 3 State price vector and risk neutral probabilities
- 4 Physical Probability v.s. Risk Neutral Probability
- 5 Binomial Model
- 6 Multi-period Binomial models
- 7 Pricing in Practice**
- 8 Summary

Pricing in Practice

Price of F at $t = 0$: $F^0 = \sum_{j=1}^{2^n} \Psi^j F^j$

• $\Psi^j = \prod_{t=1}^n \left(\psi_u \mathbb{I}_{\{Z_t^j=1\}} + \psi_d \mathbb{I}_{\{Z_t^j=-1\}} \right)$; • $(\psi_u, \psi_d) = e^{-r\delta} \left(\frac{e^{r\delta} - d}{(u-d)}, \frac{u - e^{r\delta}}{(u-d)} \right)$

- In practice, we only have initial value S_0 , maturity T , and interest r
- To implement Binomial pricing, we specify n , $S_{\max}$ & $S_{\min}$
 - Period length $\delta = \frac{T}{n}$
 - $S_{\max} = S_0 u^n \implies u = \left(\frac{S_{\max}}{S_0} \right)^{1/n}$
 - $S_{\min} = S_0 d^n \implies d = \left(\frac{S_{\min}}{S_0} \right)^{1/n}$
 - Derivative values $\{F^j\}$ is found from asset $\{S_0 u^k d^{n-k}\}_{k=0,\dots,n}$ at T
 - for path-dependent derivatives (e.g. barrier/asian options), need to investigate the whole path
 - for American options, need to price the derivative at each branch, and compare derivative and immediately exercise values
- The following R-codes present an example of pricing an up-and-out American option.

```

Deri=function(s){K=100;return(max(K-s, 0))} # Derivative's value

Price=function(n=3,S0=100,SU=133.1,SL=72.9, r=0.04, T=1){
  u=(SU/S0)^(1/n); d=(SL/S0)^(1/n); acc=exp(r*T/n) #acc:1-step accum factor
  pU=(acc-d)/(u-d)/acc; pD=(u-acc)/(u-d)/acc # Define psiU and phiD
  SPV=c(pU,pD) # SPV: State Price Vector
  G=rep(list(1:0),n);X=expand.grid(G);X=X[,n:1] # Generate the paths
  UD=d*((u/d)^X) # Generate u & d on paths
  P0=S0*t(apply(UD,1,cumprod)) # Generate possible prices of all paths
  PT=P0[,n] # Get possible prices at maturity
  PD=apply(as.matrix(PT),1,Deri) # Derivative prices at maturity
  #For barrier options only
  ##### eg. Up-110-and out: Keep only paths with max value less than 110
  PD=PD*(apply(P0,1,max)<110)
  #####
  for (k in n:1){ # Backward induction of Tree
    All.States=matrix(PD,2,2^(k-1)) # Stack all nodes in two rows: Up & Down
    PD=SPV*%All.States # Get prices at previous period
    #For American Option Only
    ##### # Find derivative value if exercise now
    if (k>1){PE=apply(as.matrix(P0[2^(n-k+1):(2^(n-k+1)+2^(k-1))],1,Deri)}
    if (k==1){PE=Deri(S0)}
    PD=pmax(PD,PE) # Derivative value = max(early exercise value, price)
    #####
  }
  return(PD)
}

```

Try this: Price(n=3,S0=100,SU=133.1,SL=72.9,r=0.04,T=1)

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Summary

- **Discrete time financial model**, with n assets, m scenarios, $\mathbf{S} \in \mathbb{R}^{m \times n}$, $\mathbf{F} \in \mathbb{R}^m$
 - (Replicating portfolio) $\mathbf{w} = \mathbf{S}^{-1}\mathbf{F}$. $\mathbf{S}^{-1}$ exists $\Rightarrow$ market is complete
 - (State price vector) $\boldsymbol{\psi}' = (\mathbf{S}^0)'\mathbf{S}^{-1} \in \mathbb{R}^m$
 - (RN probability) $\tilde{\boldsymbol{\psi}} \triangleq \boldsymbol{\psi}/\bar{\boldsymbol{\psi}}$ sums to 1
 - (Pricing formula) $F^0 \triangleq (\mathbf{S}^0)'\mathbf{w} = (\mathbf{S}^0)'\mathbf{S}^{-1}\mathbf{F} = \boldsymbol{\psi}'\mathbf{F} = \bar{\boldsymbol{\psi}}\tilde{\boldsymbol{\psi}}'\mathbf{F} = \bar{\boldsymbol{\psi}}\mathbb{E}_{\tilde{\boldsymbol{\psi}}}(\mathbf{F})$
- **Physical probabilities** ignores the importance of scenarios.
In absence of arbitrage, asset structure $\{\mathbf{S}^0, \mathbf{S}\}$ dictates pricing.
- **Binomial model**: $\mathbf{S}^0 = (1, s)'$ & $\mathbf{S} = \begin{pmatrix} e^{rT} & su \\ e^{rT} & sd \end{pmatrix}$. Payoff $\mathbf{F} = (F_u, F_d)'$
 - (Replicating Portfolio) $\mathbf{w} = \mathbf{S}^{-1}\mathbf{F} = \left(\frac{F_d u - F_u d}{e^{rT}(u-d)}, \frac{F_u - F_d}{s(u-d)} \right)'$
 - (State price vector) $\boldsymbol{\psi} = e^{-rT} \left(\frac{e^{rT} - d}{(u-d)}, \frac{u - e^{rT}}{(u-d)} \right)'$, $\bar{\boldsymbol{\psi}} = e^{-rT}$
 - (RN Probability) $\tilde{\boldsymbol{\psi}} = \left(\frac{e^{rT} - d}{u-d}, \frac{u - e^{rT}}{u-d} \right)'$
 - (Pricing) $F^0 = \boldsymbol{\psi}'\mathbf{F} = \frac{F_u(e^{rT} - d) + F_d(u - e^{rT})}{e^{rT}(u-d)}$
- **Multi-period** $\mathbf{S}^0 = (e^{rt_k}, S_k)'$, $\mathbf{S} = \begin{pmatrix} e^{rt_{k+1}} & S_k u \\ e^{rt_{k+1}} & S_k d \end{pmatrix}$, $\boldsymbol{\psi} = e^{-rT} \left(\frac{e^{rT} - d}{(u-d)}, \frac{u - e^{rT}}{(u-d)} \right)'$
 - (Path) $\boldsymbol{\omega}_n^j = (Z_1^j, \dots, Z_n^j)' \in \{1, -1\}^n$; $j = 1, \dots, 2^n$
 - (State price) $\boldsymbol{\Psi} = (\boldsymbol{\Psi}^1, \dots, \boldsymbol{\Psi}^{2^n})'$; $\boldsymbol{\Psi}^j = \prod_{t=1}^n \left(\psi_u \mathbb{I}_{\{Z_t^j=1\}} + \psi_d \mathbb{I}_{\{Z_t^j=-1\}} \right)$
 - (Pricing) of F at $t = 0$: $F^0 = \sum_{j=1}^{2^n} \boldsymbol{\Psi}^j F^j = \bar{\boldsymbol{\Psi}} \sum_{j=1}^{2^n} \tilde{\boldsymbol{\Psi}}^j F^j \triangleq \bar{\boldsymbol{\Psi}} \mathbb{E}_{\tilde{\boldsymbol{\Psi}}}(\mathbf{F})$