Vector and Lines

- Two vectors $\mathbf{v}$ and $\mathbf{w}$ are equal if they have the same length and the same direction.

- **Theorem**
Vector addition and scalar multiplication exhibit the following properties.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all vectors $\mathbf{u}$ and $\mathbf{v}$.
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all vectors $\mathbf{u}$.
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ for all vectors $\mathbf{u}$.
5. $1\mathbf{u} = \mathbf{u}$ for all vectors $\mathbf{u}$.
6. $a(b\mathbf{u}) = (ab)\mathbf{u}$ for all vectors $\mathbf{u}$ and scalars $a$, $b$.
7. $(a+b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ for all vectors $\mathbf{u}$ and scalars $a$, $b$.
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ for all vectors $\mathbf{u}$, $\mathbf{v}$ and scalars $a$.

- A vector is called a unit vector if its magnitude is 1.

- **Theorem**
Two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel if and only if one is a scalar multiple of the other.

**Example 1**
Consider a quadrilateral with vertices A, B, C and D in order. If the diagonals AC and BD bisect each other, show that the quadrilateral is a parallelogram.
Example 2
Let OAB be a right-angled triangle with the right angle at O. If C is the foot of the perpendicular from O to the hypotenuse, show that

\[ \overrightarrow{AC} = \frac{\|\overrightarrow{OA}\|^2}{\|\overrightarrow{AB}\|^2} \overrightarrow{AB} \]

Lines in Space
- Given a straight line, any nonzero vector that is parallel to the line is called a direction vector for the line.

Vector Equation of a Line
The line parallel to \( \mathbf{d} \neq \mathbf{0} \) through the point with position vector \( \mathbf{p}_0 \) is given by

\[ \mathbf{p} = \mathbf{p}_0 + t\mathbf{d} \quad \text{for some scalar } t \]

In other words, the point with position vector \( \mathbf{p} \) is on this line if and only if a real number \( t \) exists such that \( \mathbf{p} = \mathbf{p}_0 + t\mathbf{d} \).

Parametric Equations of a Line
The line through \( P_0(x_0, y_0, z_0) \) with direction vector \( \mathbf{d} = (a, b, c) \neq \mathbf{0} \) is given by

\[ x = x_0 + ta \]
\[ y = y_0 + tb \]
\[ z = z_0 + tc \]

where \( t \) is any scalar.

In other words, the point \( P(x_0, y_0, z_0) \) is on this line if and only if a real number \( t \) exists such that \( x = x_0 + ta \), \( y = y_0 + tb \) and \( z = z_0 + tc \).
Example 3
Find the vector and parametric equations of the following line: The line passing through $P(1,0,−3)$ and parallel to the line with parametric equations $x = −1 + 2t$, $y = 2 − t$ and $z = 3 + 3t$.

The Dot Product
- **Theorem**
  Let $\mathbf{v}_1 = (x_1, y_1, z_1)$ and $\mathbf{v}_2 = (x_2, y_2, z_2)$ be two vectors given in component form. Then their dot product can be computed as follows:

  $$\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1x_2 + y_1y_2 + z_1z_2$$

- $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| ||\mathbf{v}||}$

- **Theorem**
  Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example 4
Let A and B be the end points of a diameter of a circle. If C is any point on the circle, show that AC and BC are perpendicular.
Projections
- Theorem
Let \( u \) and \( d \neq 0 \) be vectors.

1. The projection \( u_1 \) of \( u \) on \( d \) is given by
   \[
   \text{proj}_d u = \frac{u \cdot d}{\|d\|^2} d.
   \]

2. The vector \( u - \text{proj}_d u \) is orthogonal to \( d \).

Example 5
Write \( u = u_1 + u_2 \), where \( u_1 \) is parallel to \( v \) and \( u_2 \) is orthogonal to \( v \).

\[
\begin{align*}
u &= (2, -1, 1), \quad v = (1, -1, 3)
\end{align*}
\]