§ 2.2.3 The General Itô Stochastic Integral

We have learnt that if \( C \) is a simple process, then \( \Gamma(C) = \int_0^t C_s \, dB_s \) is simply the Riemann–Stieltjes Sum.

We shall learn that the Itô Stochastic Integral can be defined for \( C \) satisfying

**Assumptions on the Integrand Process \( C \):**

- \( C \) is adapted to Brownian motion on \([0, T]\), i.e. \( C_t \) is a function of \( B_s, s \leq t \).
- The integral \( \int_0^T E C_s^2 \, ds \) is finite.

These assumptions are satisfied for a simple process. If \( C(t), t \in [0, T] \) is a deterministic function with \( \int_0^T C^2(t) \, dt < \infty \), then it satisfies the assumptions.

The Itô stochastic integration can be defined for the above \( C \) since

Let \( C \) be a process satisfying the Assumptions. Then one can find a sequence \((C^{(n)})\) of simple processes such that

\[
\int_0^T E[C_s - C_s^{(n)}]^2 \, ds \to 0.
\]
Indeed, one can show the existence of a process $I(C)$ on $[0, T]$ such that

$$E \sup_{0 \leq t \leq T} \left[ I_t(C) - I_t(C^{(n)}) \right]^2 \to 0.$$ 

See Appendix A4 for a proof.

The mean square limit $I(C)$ is called the \textit{Itô stochastic integral of} $C$. It is denoted by

$$I_t(C) = \int_0^t C_s \, dB_s, \quad t \in [0, T].$$

For a simple process $C$, the Itô stochastic integral has the Riemann–Stieltjes sum representation (2.11).

After this general definition we do not feel very comfortable with the notion of Itô stochastic integral. We have lost the intuition because we are not able to write the integral $\int_0^t C_s \, dB_s$ in simple terms of Brownian motion. In particular cases we will be able to obtain explicit formulae for Itô stochastic integrals, but this requires knowledge of the Itô lemma; see Section 2.3. For our general intuition, and for practical purposes, the following rule of thumb is helpful:
The Itô stochastic integrals $I_t(C) = \int_0^t C_s \, dB_s$, $t \in [0, T]$, constitute a stochastic process. For a given partition

$$\tau_n : \quad 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T,$$

and $t \in [t_{k-1}, t_k]$, the random variable $I_t(C)$ is “close” to the Riemann–Stieltjes sum

$$\sum_{i=1}^{k-1} C_{t_{i-1}} (B_{t_i} - B_{t_{i-1}}) + C_{t_{k-1}} (B_t - B_{t_{k-1}}),$$

and this approximation is the closer (in the mean square sense) to the value of $I_t(C)$ the more dense the partition $\tau_n$ in $[0, T]$.

**Properties of the General Itô Stochastic Integral**

The general Itô stochastic integral inherits the properties of the Itô stochastic integral for simple processes, see Section 2.2.2. In general, we will not be able to prove these properties. However, some proofs can be found in Appendix A4.

The stochastic process $I_t(C) = \int_0^t C_s \, dB_s$, $t \in [0, T]$, is a martingale with respect to the natural Brownian filtration $(\mathcal{F}_t$, $t \in [0, T])$.

This results from the particular choice of the approximating Riemann–Stieltjes sums of $C$, which are evaluated at the left end points of the intervals $[t_{i-1}, t_i]$.

The Itô stochastic integral has expectation zero.

For the proof of the existence of the Itô stochastic integral the isometry property (2.14) for simple processes is essential. It remains valid in the general case.

The Itô stochastic integral satisfies the *isometry property* :

$$E \left( \int_0^t C_s \, dB_s \right)^2 = \int_0^t EC_s^2 \, ds, \quad t \in [0, T].$$

The Itô stochastic integral also has some properties in common with the Riemann and Riemann–Stieltjes integrals.
The Itô stochastic integral is linear:
For constants $c_1$, $c_2$ and processes $C^{(1)}$ and $C^{(2)}$ on $[0,T]$, satisfying the Assumptions,

$$
\int_0^t \left[ c_1 C_s^{(1)} + c_2 C_s^{(2)} \right] dB_s = c_1 \int_0^t C_s^{(1)} dB_s + c_2 \int_0^t C_s^{(2)} dB_s.
$$

The Itô stochastic integral is linear on adjacent intervals:
For $0 \leq t \leq T$,

$$
\int_0^T C_s dB_s = \int_0^t C_s dB_s + \int_t^T C_s dB_s.
$$

Finally, we state the following property:

The process $I(C)$ has continuous sample paths.

Notes and Comments
The definition of the Itô stochastic integral goes back to Itô (1942,1944) who introduced the stochastic integral with a random integrand. Doob (1953) realized the connection of Itô integration and martingale theory. Itô integration is by now part of advanced textbooks on probability theory and stochastic processes. Standard references are Chung and Williams (1990), Ikeda and Watanabe (1989), Karatzas and Shreve (1988), Øksendahl (1985) and Protter (1992). Nowadays, some texts on finance also contain a survival kit on stochastic integration; see for example Lamberton and Lapeyre (1996).

Some of the aforementioned books define the stochastic integral with respect to processes more general than Brownian motion, including processes with jumps. Moreover, the assumption $\int_0^T EC_s^2 ds < \infty$ for the existence of $\int_0^t C_s dB_s$ can be substantially relaxed.
Figure 2.2.5 1st and 3rd rows: approximations to a Brownian sample path by simple processes $C^{(n)}$ (dashed lines) assuming $n = 4, 10, 20, 40$ distinct values. Below every figure you find the path of the corresponding Itô stochastic integral $I(C^{(n)})$. The latter processes approximate the process $I_t(B) = \int_0^t B_s dB_s = 0.5(B_t^2 - t)$. Compare with Figure 2.3.1, where the corresponding sample path of $I(B)$ is drawn.
§ 2.3 The Itô Lemma

We have defined $\int_0^t G_s \, dB_s$. But for only simple $G$, we be able to calculate Itô stochastic integrals. In this section, we provide an important tool to manipulate stochastic integrations.

§ 2.3.1 The Classical Rule of Differentiation

If $b$ is differentiable, then

$$\frac{1}{2} \frac{d b^2(s)}{ds} = b(s) \frac{db(s)}{ds}$$

or

$$\frac{1}{2} \int_0^t \frac{d b^2(s)}{ds} \, ds = \frac{1}{2} (b(t)^2 - b(0)) = \int_0^t b(s) \frac{db(s)}{2} \, ds$$
However, we cannot replace \( b(.) \) with \( B \).

Since
\[
\int_0^t B_s \, dB_s = \frac{1}{2} (B_t^2 - t)
\]

so
\[
2B_t \, dB_t = d[B_t^2 - t]
\]

So the ordinary differential rule for integration does not apply.

In order to understand the stochastic chain rule, we first recall the deterministic chain rule. For simplicity, we write \( h'(t), h''(t), \) etc., for the ordinary derivatives of the function \( h \) at \( t \).

Let \( f \) and \( g \) be differentiable functions. Then the classical chain rule of differentiation holds:

\[
[f(g(s))]' = f'(g(s)) g'(s).
\] (2.17)

Because we are interested in integrals, we rewrite (2.17) in integral form, i.e., we integrate both sides of (2.17) on \([0,t]\).

The chain rule in integral form:

\[
f(g(t)) - f(g(0)) = \int_0^t f'(g(s)) g'(s) \, ds = \int_0^t f'(g(s)) \, dg(s).
\] (2.18)
The above chain rule can be interpreted by Taylor expansion:
\[ f(g(t) + dg(t)) - f(g(t)) = f'(g(t)) \, dg(t) + \frac{1}{2} f''(g(t)) \, [dg(t)]^2 + \ldots \]
Integration of \( \int [dg(t)]^2 = 0 \) (high order terms)

\[ \frac{2}{2} \]

§ 2.3.2 A Simple Version of the Itô Lemma

Replacing \( g(t) \) by \( B(t) \) in (x), we have
\[ f(B_t + dB_t) - f(B_t) = f'(B_t) \, dB_t + \frac{1}{2} f''(B_t) \, (dB_t)^2 \]

Since \( (dB_t)^2 \) can be interpreted as \( dt \)
\[ \left( \sum_{t_0 < t_1 < \cdots < t_n = T} (B(t_i) - B(t_{i-1}))^2 \rightarrow T \right) \quad \text{as mesh}(\tau_n) \rightarrow 0 \]

So we have
\[ \int_0^t df(B_s) = f(B_t) - f(B_0) = \int_0^t f'(B_s) \, dB_s + \frac{1}{2} \int_0^t f''(B_s) \, ds \]

This is a simple version of Itô Lemma.
Example 2.3.2 Choose $f(t) = t^2$, then $f'(t) = 2t$ and $f''(t) = 2$. Hence

$$B_t^2 - B_s^2 = 2\int_s^t B_x \, dB_x + \int_s^t dx$$

For $s = 0$ we obtain

$$\int_0^t B_x \, dB_x = \frac{1}{2}(B_t^2 - t)$$

If $f(t) = t^3$, then $f'(t) = 3t^2$, $f''(t) = 6t$

$$B_t^3 - B_s^3 = 3\int_s^t B_x^2 \, dB_x + 6\int_s^t B_x \, dx$$

Exercise: Calculate $B_t^n - B_s^n$ for any positive integer $n$.

Example 2.3.3 (The exponential function is not the Itô exponential)

If $f(t) = e^t$, then $f'(t) = f(t)$ and

$$f(t) - f(s) = \int_s^t f(x) \, dx.$$ 

If a process $X_t$ such that

$$X_t - X_s = \int_s^t X_x \, dB_x,$$

we will call such $X_t$ the Itô exponential.
However, \( X_t = \exp(B_t) \) is not Itô exponential

\[
e^{B_t} - e^{B_s} = \int_s^t e^{B_x} dB_x + \frac{1}{2} \int_s^t e^{B_x} dW_x
\]

We need advanced Itô lemma to obtain Itô exponential

\[\text{§ 2.3.3 Extended Version of Itô Lemma}\]

In last section, we have considered

\[
\tilde{f}(B_t + dB_t) - \tilde{f}(B_t) = \ldots
\]

let \( f(t,x) \) be a two argument function and has continuous partial derivatives of at least second order, then

\[
f(t+dt, B_t + dB_t) - f(t,B_t) = \frac{\partial}{\partial t} f(t,B_t) dt + \frac{\partial}{\partial B_t} f(t,B_t) dB_t + \frac{1}{2} \left[ \frac{\partial^2}{\partial t^2} f(t,B_t) dt^2 + 2 \frac{\partial^2}{\partial t \partial B_t} f(t,B_t) dt dB_t + \frac{\partial^2}{\partial B_t^2} f(t,B_t) (dB_t)^2 \right]
\]

where

\[
\tilde{f}_i(t,x) = \frac{\partial}{\partial x_i} f(x_1,x_2) \bigg|_{x_1=t,x_2=x}
\]

\[
\tilde{f}_{ij}(t,x) = \frac{\partial^2}{\partial x_i \partial x_j} f(x_1,x_2) \bigg|_{x_1=t,x_2=x}
\]
Since \( \sum_{0 \leq t_0 < \cdots < t_n = T} (t_i - t_{i-1})^2 \to 0 \) as mesh \( \tau_n \to 0 \)

\[
\sum_{0 \leq t_0 < \cdots < t_n = T} (t_i - t_{i-1})(B(t_i) - B(t_{i-1})) \to 0 \quad \text{mesh} \tau_n \to 0
\]

(Why?)

So the terms with \( (dt)^2 \), \( \text{let} \ dB_t \) can be ignored.

but \( (dB_t)^2 \sim dt \), Therefore

**Extension I of the Itô Lemma:**

Let \( f(t, x) \) be a function whose second order partial derivatives are continuous. Then

\[
f(t, B_t) - f(s, B_s) = \int_s^t \left[ f_1(x, B_x) + \frac{1}{2} f_{22}(x, B_x) \right] dx + \int_s^t f_2(x, B_x) dB_x, \quad s < t.
\]

(2.26)

**Example 2.3.5 (The Itô exponential)**

We learnt in Example 2.3.3 that the stochastic process \( \exp\{B_t\} \) is not the Itô exponential in the sense of (2.24). Now we choose the function

\[
f(t, x) = e^{x - 0.5 t}.
\]

Then direct calculation shows that

\[
f_1(t, x) = \frac{1}{2} f(t, x), \quad f_2(t, x) = f(t, x), \quad f_{22}(t, x) = f(t, x).
\]

An application of the above Itô lemma gives

\[
f(t, B_t) - f(s, B_s) = \int_s^t f(x, B_x) dB_x.
\]

In the sense of (2.24), \( f(t, B_t) \) is the Itô exponential. See Figure 2.3.4 for a comparison of the paths of the processes \( \exp\{B_t\} \) and \( \exp\{B_t - 0.5 t\} \). \( \Box \)