§ 2.2.2

The Itô Stochastic Integral for Simple Processes

Let \( B = (B_t, t \geq 0) \) be a Brownian Motion

\[
\mathcal{F}_t = \sigma(B_s, s \leq t).
\]

The stochastic process \( C = (C_t, t \in [0, T]) \) is said to be simple if it satisfies the following properties:

There exists a partition

\[
\tau_n : \quad 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T,
\]

and a sequence \( (Z_i, i = 1, \ldots, n) \) of random variables such that

\[
C_t = \begin{cases} 
Z_n, & \text{if } t = T, \\
Z_i, & \text{if } t_{i-1} \leq t < t_i, \quad i = 1, \ldots, n.
\end{cases}
\]

The sequence \( (Z_i) \) is adapted to \( (\mathcal{F}_{t_{i-1}}, i = 1, \ldots, n) \), i.e. \( Z_i \) is a function of Brownian motion up to time \( t_{i-1} \), and satisfies \( EZ_i^2 < \infty \) for all \( i \).

**Example 2.2.1** (Some simple processes)

The deterministic function

\[
f_n(t) = \begin{cases} 
\frac{n-1}{n}, & \text{if } t = T, \\
\frac{i-1}{n}, & \text{if } \frac{i-1}{n} \leq t < \frac{i}{n}, \quad i = 1, \ldots, n,
\end{cases}
\]

is a step function, hence it is a simple function.

Now define the process

\[
C_t = \begin{cases} 
Z_n = B_{t_{n-1}}, & \text{if } t = T, \\
Z_i = B_{t_{i-1}}, & \text{if } t_{i-1} \leq t < t_i, \quad i = 1, \ldots, n,
\end{cases}
\]

for a given partition \( \tau_n \) of \([0, T]\). It is a simple process: the paths are piecewise constant, and \( C_t \) is a function of Brownian motion until time \( t \). For an illustration of the process \( C \) for two different partitions, see Figure 2.2.2.
Figure 2.2.2 Two approximations of a Brownian sample path by simple processes $C$ (dashed lines), given by (2.10).

The Itô stochastic integral of a simple process $C$ on $[0, t]$, $t_{k-1} \leq t \leq t_k$, is given by

$$
\mathcal{I}_t(C) \triangleq \int_0^t C_s \, dB_s :=
$$

$$
\int_0^T C_s \, I_{[0,t]}(s) \, dB_s = \sum_{i=1}^{k-1} Z_i \Delta_i B + Z_k (B_t - B_{t_{k-1}}), \quad (2.11)
$$

where $\sum_{i=1}^0 Z_i \Delta_i B = 0$.

Thus the value of the Itô stochastic integral $\int_0^t C_s \, dB_s$ is the Riemann–Stieltjes sum of the path of $C$, evaluated at the left end points of the intervals $[t_{i-1}, t_i]$, with respect to Brownian motion. If $t < t_n$, the point $t$ can formally be interpreted as the last point of the partition of $[0, t]$. 
Figure 2.2.3 The Itô stochastic integral $\int_0^t C_s \, dB_s$ corresponding to the paths of $C$ and $B$ given in Figure 2.2.2.

Example 2.2.4 (Continuation of Example 2.2.1)
Recall the simple processes $f_n$ and $C$ from Example 2.2.1. The corresponding Itô stochastic integrals are then given by

$$\int_0^t f_n(s) \, dB_s = \sum_{i=1}^{k-1} \frac{i-1}{n} \left( B_{i/n} - B_{(i-1)/n} \right) + \frac{k-1}{n} \left( B_t - B_{(k-1)/n} \right),$$

for $t \in [(k-1)/n, k/n]$, and by

$$\int_0^t C_s \, dB_s = \sum_{i=1}^{k-1} B_{t_{i-1}} \Delta_i B + B_{t_{k-1}} \left( B_t - B_{t_{k-1}} \right),$$

for $t \in [t_{k-1}, t_k]$.

We need to define $\int_0^t f(s) \, dB_s$ for general $f$. Towards this let us first look at the properties of $I(C)$.

The stochastic process $I_t(C) = \int_0^t C_s \, dB_s$, $t \in [0, T]$, is a martingale with respect to the natural Brownian filtration ($\mathcal{F}_t$, $t \in [0, T]$).
We need to check

- $E[I_t(G)] < \infty$ for all $t \in [0, T]$
- $I_t(G)$ is adapted to $(\mathcal{F}_t)$

$$E(I_t(G)|\mathcal{F}_s) = I_s(G) \quad \text{for } s < t \quad (*)$$

The first two "\*" is easy to see. For $(*), let us first assume that $s < t$, $s, t \in [t_{k-1}, t_k]$. 

$$I_t(G) = I_{t_{k-1}}(G') + Z_k(B_t - B_{t_{k-1}})$$

$$= I_{t_{k-1}}(G) + Z_k(B_s - B_{t_{k-1}}) + Z_k(B_t - B_s)$$

$$= I_t(G) + Z_k(B_t - B_s)$$

Therefore

$$E[I_t(G)|\mathcal{F}_s] = I_s(G) + E[Z_k(B_t - B_s)|\mathcal{F}_s]$$

$$= I_s(G) + Z_k E[B_t - B_s|\mathcal{F}_s] = I_s(G')$$

Since $I_s(G')$ is adapted to $\mathcal{F}_s$ and $Z_k$ is adapted to $\mathcal{F}_s$

Next assume that $s \in [t_{k-1}, t_k], t \in [t_{k-1}, t_k], l < k$.

$$I_t(G) = I_{t_{k-1}}(G) + Z_l(B_t - B_{t_{k-1}})$$

$$+ [Z_l(B_{t_l} - B_s) + \sum_{k=1}^{k-1} Z_k \Delta B + Z_k(B_t - B_{t_{k-1}})]$$

We may show $E[\ldots |\mathcal{F}_s] = 0$ \ldots
Properties of the Itô Stochastic Integral for Simple Processes

1. The Itô stochastic integral has expectation zero.

2. The Itô stochastic integral satisfies the isometry property:

\[ E \left( \int_0^t C_s \, dB_s \right)^2 = \int_0^t EC_s^2 \, ds, \quad t \in [0, T]. \]

Proof of 2. Assume that \( 0 = t_0 < t_1 < \ldots < t_k = t \)

is the partition restricted to \([0, t]\). Then

\[ E[I_t^2] = \sum_{i=1}^{k} \sum_{j=1}^{k} E[W_i W_j], \]

where \( W_i = Z_i(B_{t_i} - B_{t_{i-1}}), \ i=1, \ldots, k. \)

It is not hard to see that

\[ EW_i W_j = 0 \text{ if } i \neq j, \]

\[ EW_i W_j = E[Z_i^2] E[(B_{t_i})^2] = E[Z_i^2] E[B_{t_i}^2] = E[B_{t_i}^2]. \]

if \( i = j. \)

Therefore

\[ E[I_t^2] = \sum_{i=1}^{k} E[Z_i^2] (t_i - t_{i-1}) = \int_0^t EC_s^2 \, ds. \]
The Itô stochastic integral is linear:
For constants $c_1$, $c_2$ and simple processes $C^{(1)}$ and $C^{(2)}$ on $[0, T],$

$$\int_0^t \left[ c_1 C^{(1)}_s + c_2 C^{(2)}_s \right] dB_s = c_1 \int_0^t C^{(1)}_s dB_s + c_2 \int_0^t C^{(2)}_s dB_s.$$

The Itô stochastic integral is linear on adjacent intervals:
For $0 \leq t \leq T,$

$$\int_0^T C_s dB_s = \int_0^t C_s dB_s + \int_t^T C_s dB_s.$$

The first of 3 is by the observation that if $C_1$ and $C_2$ are simple, then $c_1 C_1 + c_2 C_2$ is simple.

The second is straight from definition.

4. The process $I(C)$ has continuous sample paths.

This is also straight from definition.

In general we may view that the $I(C)$ is a non-homogeneous Brownian motion.