The Jensen Inequality:

Let \( f \) be a convex function on \( \mathbb{R} \). If \( E|X| \) and \( E|f(X)| \) are finite, then

\[
f(E(X|\mathcal{F})) \leq E(f(X)|\mathcal{F}) \quad \cdots (x)
\]

Convex function: For every \( x \), there exist \( a_x \) such that

\[
f(y) \geq f(x) + a_x (y-x) \quad \cdots (**)\]

for all \( y \).

- In particular, if \( f(y) \) is twice differentiable and \( f''(y) \geq 0 \), then \( f \) is convex.

**Proof** By Taylor expansion,

\[
f(y) = f(x) + f'(x)(y-x) + \frac{f''(y_x)}{2}(y-x)^2
\]
Proof of Jensen Inequality

From (**) we have

\[ f(x) \geq f(\mathbb{E}[X|\mathcal{F}]) + a_{\mathbb{E}[X|\mathcal{F}]}(\mathbb{E}[X|\mathcal{F}] - x) \]

Taking conditional expectation of \( \mathbb{E} \cdot | \mathcal{F} \) and contains since \( a_{\mathbb{E}[X|\mathcal{F}]} \) is a function no more information than \( \mathcal{F} \),

\[ \mathbb{E}[a_{\mathbb{E}[X|\mathcal{F}]}(\mathbb{E}[X|\mathcal{F}] - x)|\mathcal{F}] = a_{\mathbb{E}[X|\mathcal{F}]}(\mathbb{E}[X|\mathcal{F}] - \mathbb{E}[X|\mathcal{F}]) = 0 \]

We readily have (**).

Consequence of Jensen Inequality

1. \( \mathbb{E}[\mathbb{E}[X|\mathcal{F}]] \leq \mathbb{E}[\mathbb{E}[X|\mathcal{F}]] = \mathbb{E}[X] \)

2. If \( 0 < q < p \), then

\[ \mathbb{E}(X^q)^{1/q} \leq (\mathbb{E}|X|^p)^{1/p} \]

or

\[ \mathbb{E}|X| \leq (\mathbb{E}|X|^2)^{1/2} \quad (\text{with } q = 1, p = 2) \]
A3 Non-Differentiability and Unbounded Variation of Brownian Sample Paths

Let $B = (B_t, t \geq 0)$ be Brownian motion. Recall the definitions of an $H$-self-similar process from (1.12) on p. 36 and of a process with stationary increments from p. 30. We also know that Brownian motion is a $0.5$-self-similar process with stationary, independent increments; see Section 1.3.1. We show the non-differentiability of Brownian sample paths in the more general context of self-similar processes.

**Proposition A3.1** (Non-differentiability of self-similar processes)

Suppose $(X_t)$ is $H$-self-similar with stationary increments for some $H \in (0, 1)$. Then for every fixed $t_0$,

$$\limsup_{t \downarrow t_0} \frac{|X_t - X_{t_0}|}{t - t_0} = \infty,$$

i.e. sample paths of $H$-self-similar processes are nowhere differentiable with probability 1.

**Proof.** Without loss of generality we choose $t_0 = 0$. Let $(t_n)$ be a sequence such that $t_n \downarrow 0$. Then, by $H$-self-similarity, $X_0 = 0$ a.s., and hence

$$P \left( \lim_{n \to \infty} \sup_{0 \leq s \leq t_n} \frac{|X_s|}{s} > x \right) = \lim_{n \to \infty} P \left( \sup_{0 \leq s \leq t_n} \frac{|X_s|}{s} > x \right)$$

$$\geq \limsup_{n \to \infty} P \left( \frac{X_{t_n}}{t_n} > x \right)$$

$$= \limsup_{n \to \infty} P \left( t_n^{H-1} |X_1| > x \right)$$

$$= 1, \quad x > 0.$$

Hence, with probability 1, $\limsup_{n \to \infty} |X_{t_n}/t_n| = \infty$ for any sequence $t_n \downarrow 0$. \qed
Proposition A3.2 (Unbounded variation of Brownian sample paths)

For almost all Brownian sample paths,

\[ v(B(\omega)) = \sup_{\tau} \sum_{i=1}^{n} |B_{t_i}(\omega) - B_{t_{i-1}}(\omega)| = \infty \text{ a.s.}, \]

where the supremum is taken over all possible partitions \( \tau : 0 = t_0 < \cdots < t_n = T \) of \([0, T]\).

Proof. For convenience, assume \( T = 1 \). Suppose that \( v(B(\omega)) < \infty \) for a given \( \omega \). Let \( (\tau_n) \) be a sequence of partitions \( \tau_n : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1 \), such that \( \text{mesh}(\tau_n) \to 0 \). Recall that \( \Delta_i B = B_{t_i} - B_{t_{i-1}} \). The following chain of inequalities holds:

\[
Q_n(\omega) = \sum_{i=1}^{n} (\Delta_i B(\omega))^2 \\
\leq \max_{i=1,\ldots,n} |\Delta_i B(\omega)| \sum_{i=1}^{n} |\Delta_i B(\omega)| \\
\leq \max_{i=1,\ldots,n} |\Delta_i B(\omega)| \cdot v(B(\omega)). \tag{A.3}
\]

Since \( B \) has continuous sample paths with probability 1, we may assume that \( B_t(\omega) \) is a continuous function of \( t \). It is also uniformly continuous on \([0, 1]\) which, in combination with \( \text{mesh}(\tau_n) \to 0 \), implies that \( \max_{i=1,\ldots,n} |\Delta_i B(\omega)| \to 0 \). Hence the right-hand side of (A.3) converges to zero, implying that

\[ Q_n(\omega) \to 0. \tag{A.4} \]

On the other hand, we know from p. 98 that \( Q_n \xrightarrow{P} 1 \), hence \( Q_{n_k}(\omega) \xrightarrow{a.s.} 1 \) for a suitable subsequence \((n_k)\); see p. 187. Thus (A.4) is only possible on a null-set, and so

\[ P(\{\omega : v(B(\omega)) = \infty\}) = 1. \]

\( \Box \)