§ 4.2.2 An interpretation of the Black–Scholes Formula by Change of Measure

The Black–Scholes option pricing formula can be interpreted as the expectation of the discounted overshoot $(X_T - K)^+$. 

The Black–Scholes Model

- The price of one share of the risky asset (stock) is described by the stochastic differential equation

$$dX_t = cX_t\, dt + \sigma X_t\, dB_t, \quad t \in [0, T],$$

where $c$ is the mean rate of return, $\sigma$ the volatility, $B$ is standard Brownian motion (under $P$) and $T$ is the time of maturity of the option.

- The price of the riskless asset (bond) is described by the deterministic differential equation

$$d\beta_t = r \beta_t\, dt, \quad t \in [0, T],$$

where $r > 0$ is the interest rate of the bond.

- Your portfolio at time $t$ consists of $a_t$ shares of stock and $b_t$ shares of bond. Thus its value at time $t$ is given by

$$V_t = a_t X_t + b_t \beta_t, \quad t \in [0, T].$$

- The portfolio is self-financing, i.e.

$$dV_t = a_t\, dX_t + b_t\, d\beta_t, \quad t \in [0, T].$$

- At time of maturity, $V_T$ is equal to the contingent claim $h(X_T)$ for a given function $h$. For a European call option, $h(x) = (x - K)^+$, where $K$ is the strike price of the option, and for a European put option, $h(x) = (K - x)^+$. 

We shall change the probability measure $P$ to $Q$ in such a way that the discounted price of one share of stock
\[ \widetilde{X}_t = e^{-rt} X_t, \quad t \in [0,T] \]
will become a martingale. Write
\[ f(t,x) = e^{-rt} \cdot x \]
Apply Itô lemma,
\begin{align*}
\frac{d\widetilde{X}_t}{\widetilde{X}_t} &= -re^{-rt} X_t \, dt + e^{-rt} \frac{dX_t}{X_t} \\
&= -re^{-rt} X_t \, dt + e^{-rt} X_t \left[ cdt + \sigma dB_t \right] \\
&= \widetilde{X}_t \left[ (c-r)dt + \sigma dB_t \right] \\
&\triangleq \sigma \widetilde{X}_t \, dB_t \\
&\quad \ldots \ldots (\ast)
\end{align*}
Where
\[ \widetilde{B}_t = B_t + \left[ \frac{(C-r)\varnothing}{\sigma} \right] t \]
From Girsanov’s theorem we know that there exists an equivalent martingale measure $Q$ which turn $\widetilde{B}$ into standard Brownian motion.
The solution to (25) is
\[ \bar{X}_t = X_0 \, e^{-0.5 \sigma^2 t + \sigma \bar{B}_t} \]
which is a martingale under probability measure \( Q \).

Assume in the Black–Scholes model that there exists a self-financing strategy \((a_t, b_t)\) such that the value \( V_t \) of your portfolio at time \( t \) is given by
\[ V_t = a_t X_t + b_t \beta_t, \quad t \in [0, T], \]
and that \( V_T \) is equal to the contingent claim \( h(X_T) \).
Then the value of the portfolio at time \( t \) is given by
\[ V_t = E_Q \left[ e^{-r(T-t)} h(X_T) \mid \mathcal{F}_t \right], \quad t \in [0, T], \tag{4.29} \]
where \( E_Q(A \mid \mathcal{F}_t) \) denotes the conditional expectation of the random variable \( A \), given \( \mathcal{F}_t = \sigma(B_s, s \leq t) \), under the new probability measure \( Q \).

**Proof of (4.29)** Consider the discounted value
\[
\begin{align*}
V_t &= e^{-rt} V_t = e^{-rt} (a_t X_t + b_t \beta_t) \\
\frac{d \bar{V}_t}{V_t} &= -r \bar{V}_t \, dt + e^{-rt} \, d \bar{V}_t \quad (\text{Ito's lemma}) \\
&= -re^{-rt} (a_t X_t + b_t \beta_t) \, dt + e^{-rt} (a_t dX_t + b_t d\beta_t) \\
&= a_t (-re^{-rt} X_t \, dt + e^{-rt} \, dX_t) \\
&= a_t \, dX_t.
\end{align*}
\]
Since $\tilde{V}_0 = V_0$, we have

$$\tilde{V}_t = V_0 + \int_0^t a_s dX_s = V_0 + \int_0^t a_s \tilde{X}_s d\tilde{B}_s$$

So $\tilde{V}_t$ is a martingale with respect to $(\mathcal{F}_t, t \in [0,T])$, where $\mathcal{F}_t = \sigma(B_s, s \leq t)$ is the natural filtration.

By Martingale property,

$$\tilde{V}_t = \mathbb{E}_Q(\tilde{V}_T | \mathcal{F}_t), \quad t \in [0,T]$$

but $\tilde{V}_T = e^{-rT} V_T = e^{-rT} \mathcal{L}(X_T)$.

Hence $e^{-rt} \tilde{V}_t = \mathbb{E}_Q( e^{-rT} \mathcal{L}(X_T) | \mathcal{F}_t)$,

which is equivalent to (4.29).

**Example** (The value of a European option)

By (4.29), the value $V_t$ of the portfolio at time $t$ corresponding to the contingent claim $V_T = \mathcal{L}(X_T)$ is given by

$$V_t = \mathbb{E}_Q[ e^{-r\theta} \mathcal{L}(X_T) | \mathcal{F}_t ], \quad \theta = T-t,$$

$$= \mathbb{E}_Q[ e^{-r\theta} \mathcal{L}(X_t e^{(r-\sigma^2/2)\theta + \sigma(B_t - \tilde{B}_t)}) | \mathcal{F}_t ]$$

At time $t$, $X_t$ can be treated as constant. $\tilde{B}_t - B_t$ is independent of $\mathcal{F}_t$ and has $N(0, \theta)$ distribution.
Therefore
\[ V_t = e^{-r_0} \int_{-\infty}^{\infty} \psi(X_t e^{(r-0.5\sigma^2)t + \sigma Y \theta^{1/2}}) \phi(y) \, dy, \]
\( \phi(y) \) is the standard normal density. For a European call option,
\[ h(x) = (x-K)^+ = \max(0, x-K) \]
\[ V_t = \int_{y: y > \frac{\ln(x/k) + (r+0.5\sigma^2)t}{\sigma \theta^{1/2}} + \sigma \theta^{1/2}} \left( X_t e^{(r-0.5\sigma^2)t + \sigma Y \theta^{1/2}} - K \right) \phi(y) \, dy \]
\[ = \int_{y: y > -\frac{\ln(x/k) + (r+0.5\sigma^2)t}{\sigma \theta^{1/2}} + \sigma \theta^{1/2}} \phi(y) \, dy \]
\[ = X_t \Phi(z_1) - Ke^{-r\theta} \Phi(z_2) \]
Where \( \Phi(\cdot) \) is the standard normal distribution function
\[ z_1 = \frac{\ln(x/k) + (r+0.5\sigma^2)t}{\sigma \theta^{1/2}}, \quad z_2 = z_1 - \sigma \theta^{1/2} \]
For \( t = 0 \), this is exactly the Black-Scholes option price for a European Call option!