4.1.3. Build up a Self-financing Strategy
to achieve return $(X_T - K)^+$ at time $t = T$.

In this subsection we try to derive a Self-financing
strategy to achieve $(X_T - K)^+$ at time $t = T$.

Suppose $(a_t, b_t)$ is such a strategy with
value process

$$V_t = a_t X_t + b_t \beta_t \triangleq u(T-t, X_t), t \in [0, T]$$

for some smooth deterministic function $u(t, x)$.

(Later on we shall know that such function $u$ do
exist.). The exact form of $u(t, x)$ is arbitrary
but it has to satisfy

$$V_T = u(0, X_T) = (X_T - K)^+.$$ 

This is also called **hedging against the contingent claim** $(X_T - K)^+$.

We intend to use Itô's lemma to derive the
exact form of $u$ and $a_t, b_t$. 

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Write \( f(t, x) = u(T-t, x) \), then

\[
\begin{align*}
\hat{f}_1(t, x) &= -u_1(T-t, x), & \hat{f}_2(t, x) &= u_2(T-t, x), \\
\hat{f}_{22}(t, x) &= u_{22}(T-t, x)
\end{align*}
\]

Also recall that \( X \) is a Itô process with

\[
X_t = X_0 + c \int_0^t X_s \, ds + \sigma \int_0^t X_s \, dB_s.
\]

Now apply Itô lemma we have

\[
V_t - V_0 = \hat{f}(t, X_t) - \hat{f}(0, X_0)
\]

\[
= \int_0^t \left[ \hat{f}_1(s, X_s) + \frac{\sigma^2}{2} X_s \hat{f}_{22}(s, X_s) \right] \, ds
\]

\[
+ \int_0^t \hat{f}_2(s, X_s) \, dX_s
\]

\[
= \int_0^t \left[ -u_1(T-s, X_s) + c X_s u_2(T-s, X_s) + \frac{\sigma^2}{2} X_s u_{22}(T-s, X_s) \right] \, ds
\]

\[
+ \int_0^t \sigma X_s u_2(T-s, X_s) \, dB_s
\]

\[
\vdots \quad \text{(1)}
\]
V_t - V_0 = \int_0^t a_s dX_s + \int_0^t b_s dB_s

Since \beta_t = \beta_0 e^{rt}, \quad dB_t = r\beta_0 e^{rt} dt = r\beta_t dt

Moreover, \quad V_t = a_t X_t + b_t \beta_t, \quad thus

b_t = \frac{V_t - a_t X_t}{\beta_t}

Combining above, we obtain

V_t - V_0 = \int_0^t a_s dX_s + \int_0^t \frac{V_s - a_s X_s}{\beta_s} r\beta_s dB_s

= \int_0^t a_s dX_s + \int_0^t r(V_s - a_s X_s) ds

= \int_0^t ca_s X_s ds + \int_0^t \sigma a_s X_s dB_s + \int_0^t r(V_s - a_s X_s) ds

= \int_0^t [(ca_s - \sigma a_s) X_s + rV_s] ds + \int_0^t \sigma a_s X_s dB_s \quad \ldots (..)

Compare (x) with (y), we obtain

a_t = u_2(T-t, X_t)

(c-r)a_t X_t + ru(T-t, X_t) = (c-r) u_2(T-t, X_t) X_t + ru(T-t, X_t)

= -u_1(T-t, X_t) + cX_t u_2(T-t, X_t) + a5\sigma^2 X_t^2
The above equation can be written as

\[ u_1(t,x) = 0.5 \sigma^2 x^2 u_{22}(t,x) + r x u_2(t,x) - r u(t,x) \quad \text{if} \quad x > 0, \ t \in [0, T] \]

Recalling the terminal condition, we have

\[ V_T = u(0, x_T) = (x_T - K)^+ \]

which reduces to the restriction on \( u(0,x) \) as

\[ u(0,x) = (x - K)^+, \quad x > 0. \quad \ldots \quad (*) \]

§ 4.1.4 The Black and Scholes Formula

The partial differential equation (*) has nice explicit solution:

\[ u(t,x) = x \Phi(g(t,x)) - K e^{-rt} \Phi(hlt,x)) \]

where

\[ g(t,x) = \frac{\ln(x/K) + (r + 0.5 \sigma^2) t}{\sigma t^{1/2}} \]

\[ h(t,x) = \frac{g(t,x) - \sigma t^{1/2}}{\sigma t^{1/2}} \]

and

\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \quad x \in \mathbb{R}. \]

Exercise: Verify that \( u(t,x) \) given above is a solution to (*) with initial condition (*).
What did we actually do?

\[ V_0 = u(T, X_0) = X_0 \Phi(q(T, X_0)) - Ke^{-rT} \Phi(L(T, X_0)) \]

is a rational price at time \( t=0 \) for a European call option with exercise price \( K \).

The stochastic process \( V_t = u(T-t, X_t) \) is the value of your self-financing portfolio at time \( t \in [0, T] \).

The self-financing strategy \((a_t, b_t)\) is given by

\[
\begin{align*}
a_t &= u_2(T - t, X_t) \quad \text{and} \quad b_t = \frac{u(T - t, X_t) - a_t X_t}{\beta_t},
\end{align*}
\]

(4.14)

see (4.10) and (4.8).

At time of maturity \( T \), the formula (4.13) yields the net portfolio value of \((X_T - K)^+\). Moreover, one can show that \( a_t > 0 \) for all \( t \in [0, T] \), but \( b_t < 0 \) is not excluded. Thus short sales of stock do not occur, but borrowing money at the bond’s constant interest rate \( r > 0 \) may become necessary.

Equation (4.13) is the celebrated Black-Scholes option pricing formula. We see that it is independent of the mean rate of return \( c \) for the price \( X_t \), but it depends on the volatility \( \sigma \).
Why is $u(T, X_0)$ a rational price?

If we want to understand $q = u(T, X_0)$ as a rational value in terms of arbitrage, suppose that the initial option price $p \neq q$. If $p > q$, apply the following strategy: at time $t = 0$

- sell the option to someone else at the price $p$, and
- invest $q$ in stock and bond according to the self-financing strategy (4.14).

Thus you gain an initial net profit of $p - q > 0$. At time of maturity $T$, the portfolio has value $a_T X_T + b_T \beta_T = (X_T - K)^+$, and you have the obligation to pay the value $(X_T - K)^+$ to the purchaser of the option. This means: if $X_T > K$, you must buy the stock for $X_T$, and sell it to the option holder at the exercise price $K$, for a net loss of $X_T - K$. If $X_T \leq K$, you do not have to pay anything, since the option will not be exercised. Thus the total terminal profit is zero, and the net profit is $p - q$.

The scale of this game can be increased arbitrarily, by selling $n$ options for $np$ at time zero and by investing $nq$ in stock and bond according to the self-financing strategy $(n a_t, n b_t)$. The net profit will be $n (p - q)$. Thus the opportunity for arbitrarily large profits exists without an accompanying risk of loss. This means arbitrage. Similar arguments apply if $q > p$; now the purchaser of the option will make arbitrarily big net profits without accompanying risks.

In 1997, Merton & Scholes were awarded the Nobel prize for economics for the work we showed in this chapter.
§ 4.2 A Useful Technique: Change of Measure

Change of Measure, A Example

A random variable \( X \) with density function \( q \) is usually defined for a given probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

Under measure \( \mathbb{P} \),

\[
\mathbb{P}\{X \in (a,b]\} = \int_a^b q(x) \, dx
\]

However, if we change the measure \( \mathbb{P} \) to \( Q \) where

\[
Q(A) = \int_A f(w) \, d\mathbb{P}(w),
\]

Then under \( Q \), the random variable may not have \( Q \) as its density. In fact, we have

\[
Q\{X \in (a,b]\} = \int_a^b f_i(x) q_i(x) \, dx
\]

\( f_i(x) q_i(x) \) may or may not be a density function.
Let $P$ and $Q$ be two probability measures on the $\sigma$-field $\mathcal{F}$. If there exists a non-negative function $f_1$ such that

$$Q(A) = \int_A f_1(\omega) \, dP(\omega), \quad A \in \mathcal{F},$$

we say that $f_1$ is the density of $Q$ with respect to $P$ and we also say that $Q$ is absolutely continuous with respect to $P$.

The integrals in (4.17) have to be interpreted in the measure-theoretic sense.

In a similar way, changing the roles of $P$ and $Q$, we can introduce the density $f_2$ of $P$ with respect to $Q$, given such a non-negative function exists.

If $P$ is absolutely continuous with respect to $Q$, and $Q$ is absolutely continuous with respect to $P$, we say that $P$ and $Q$ are equivalent probability measures.

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Let $B = (B_t, t \in [0, T])$ be a standard Brownian motion defined on it.

We are interested in the process of the form

$$\tilde{B}_t = B_t + \theta_t, \quad t \in [0, T]$$

for some constant $\theta$. With the only exception when $\theta = 0$, $\tilde{B}$ is not standard Brownian motion. However, if we change the underlying probability measure $P$ for an appropriate probability measure $Q$, $\tilde{B}$ can be a standard Brownian motion.
Let
\[ \mathcal{F}_t = \sigma(B_s, s \leq t), \quad t \in [0, T], \]  
(4.19)
is the Brownian filtration.

**Girsanov’s Theorem:**
The following statements hold:

- The stochastic process
  \[ M_t = \exp \left\{ -qB_t - \frac{1}{2} q^2 t \right\}, \quad t \in [0, T], \]  
  (4.20)
is a martingale with respect to the natural Brownian filtration (4.19) under the probability measure \( P \).

- The relation
  \[ Q(A) = \int_A M_T(\omega) \, dP(\omega), \quad A \in \mathcal{F}, \]  
  (4.21)
defines a probability measure \( Q \) on \( \mathcal{F} \) which is equivalent to \( P \).

- Under the probability measure \( Q \), the process \( \tilde{B} \) defined by (4.18) is a standard Brownian motion.

- The process \( \tilde{B} \) is adapted to the filtration (4.19).

The probability measure \( Q \) is called an *equivalent martingale measure*.

The change of measure serves the purpose of eliminating the drift term in a stochastic differential equation.
Example 4.2.1 (Elimination of the drift in a linear stochastic differential equation)

Consider the linear stochastic diff. equation

\[ dX_t = aX_t \, dt + \sigma X_t \, dB_t \quad \cdots (\ast) \]

Introduce \( \widehat{B}_t = B_t + (\mu \sigma) t, \quad t \in [0, T] \)

Then (\ast) becomes

\[ dX_t = \sigma X_t \, d\widehat{B}_t \quad \cdots (\ast\ast) \]

Under measure \( Q, \) (with \( M_t(\omega) = \exp \left\{ -\frac{8B_t - \frac{1}{2} \sigma^2 t}{2} \right\} \))

\( \widehat{B}_t \) is a Brownian motion \( \mu = (\mu \sigma) \)

Therefore, we can solve (\ast\ast). The solution is

\[ X_t = X_0 \exp \left\{ -\frac{1}{2} \sigma^2 t + \sigma \widehat{B}_t \right\} \quad \text{(simpler !!!)} \]

Going back to \( B_t, \) we have

\[ X_t = X_0 \exp \left\{ (c - 0.5 \sigma^2) t + \sigma B_t \right\} \]

as a solution to (\ast).