Chapter 4 Applications of Stochastic Calculus in Finance

Applications Modelling prices of risky assets
Modelling movements of small particles.

§ 4.1 The Black-Scholes Option Pricing Formula

§ 4.1.1. Introduction to Finance

Suppose that \( X_t \) is a risky assets. It satisfies

\[
dX_t = cX_t \, dt + \sigma X_t \, dB_t \quad \cdots \quad (*)
\]

We know that the unique strong solution to \((*)\) is

\[
X_t = X_0 \exp \left\{ (c - \frac{1}{2} \sigma^2) t + \sigma B_t \right\}
\]

If we imagine \( dt \) as a small increment of \( t \), then \((*)\) can be interpreted by

\[
\frac{X_{t+dt} - X_t}{X_t} = c \, dt + \sigma \, dB_t
\]
So solution to (**) can be used as a first approximation to a stock price.

\[ C \quad \text{mean rate of return} \]

\[ \sigma \quad \text{volatility} \]

Now suppose we also have a non-risky asset, a bond, which has interest rate \( r \). Let the initial investment is \( \beta_0 \), then at time \( t \), the bond worth

\[ \beta_t = \beta_0 e^{rt} \]

or it satisfies

\[ d\beta_t = r\beta_t dt \]

Portfolio — \( a_t \) shares in \( X_t \) and \( b_t \) shares in \( \beta_t \). denoted as \((a_t, b_t)\)

Wealth: \( V_t = a_t X_t + b_t \beta_t \)

\( a_t \) and \( b_t \) may assume positive or negative values. \( a_t \) negative means short sell of stocks. \( b_t \) negative means borrow money at bond's riskless interest rate \( r \).
Assuming no transaction costs and no consumption.

A trading strategy \((a_t, b_t)\) is self-financing if the increments of wealth \(V_t\) result only from changes of the prices \(X_t\) and \(\beta_t\) of your assets, or

\[
\begin{align*}
\text{or} & \quad dV_t = d(a_tX_t + b_t\beta_t) = a_t dX_t + b_t d\beta_t \\
\text{or} & \quad V_t - V_0 = \int_0^t a_s dX_s + \int_0^t b_s d\beta_s
\end{align*}
\]

We may think that the amount of \(a_t\) is decided before \(X_t\) and \(\beta_t\) are available.

\(^\text{§} 4.1.2\) Option

A European option purchased today \((t=0)\) will be worth \((X_T - K)^+\) at maturity or time of expiration \(T\). \(K\) is called the strike price of the option.

A American option with maturity \(T\) and strike price \(K\) worth \(\max_{0 \leq t \leq T} (X_t - K)^+\).
Where we have used the mathematical function

\[ a^+ = \begin{cases} a, & \text{if } a > 0 \\ 0, & \text{if } a \leq 0. \end{cases} \]

A European put option with maturity $T$ and strike price $K$ will worth

\[(K - X_T)^+\]

The purchaser of a put option may sell stock at price $K$ at time $T$. This option will only be exercised only if $K > X_T$.

We will only discuss European call option.

**Figure 4.1.1** The value of a European call option with exercise price $K$ at time of maturity $T$. 
Since at $t=0$, we do not know $X_T$, the worth of an option at time $t=0$ is an unknown random variable.

How much would you be willing to pay for such a ticket, i.e. what is a rational price for this option at time $t = 0$?

**Simple answer**: $E(X_T - K)^+$

or the expected worth at time $T$. However, a better evaluation is given by Black, Scholes and Merton.

- An individual, after investing this rational value of money in stock and bond at time $t = 0$, can manage his/her portfolio according to a self-financing strategy (see p. 169) so as to yield the same payoff $(X_T - K)^+$ as if the option had been purchased.

- If the option were offered at any price other than this rational value, there would be an opportunity of arbitrage, i.e. for unbounded profits without an accompanying risk of loss.