A strong solution to the Itô stochastic differential equation

\[ X_t = X_0 + \int_0^t a(s, X_s) \, ds + \int_0^t b(s, X_s) \, dB_s, \quad 0 \leq t \leq T \]  

is a stochastic process \( X = (X_t, t \in [0, T]) \) which satisfies the following conditions:

- \( X \) is adapted to Brownian motion, i.e. at time \( t \) it is a function of \( B_s, s \leq t \).
- The integrals occurring in (3.7) are well defined as Riemann or Itô stochastic integrals, respectively.
- \( X \) is a function of the underlying Brownian sample path and of the coefficient functions \( a(t, x) \) and \( b(t, x) \).

Existence of Strong Solutions to (3.7)

Assume the initial condition \( X_0 \) has a finite second moment: \( E X_0^2 < \infty \), and is independent of \( (B_t, t \geq 0) \).

Assume that, for all \( t \in [0, T] \) and \( x, y \in \mathbb{R} \), the coefficient functions \( a(t, x) \) and \( b(t, x) \) satisfy the following conditions:

- They are continuous.
- They satisfy a Lipschitz condition with respect to the second variable:

\[ |a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \leq K |x - y|. \]

Then the Itô stochastic differential equation (3.7) has a unique strong solution \( X \) on \([0, T]\).
Example 3.2.2 (Linear stochastic differential equation)
Consider the Itô stochastic differential equation

\[ X_t = X_0 + \int_0^t (c_1 X_s + c_2) \, ds + \int_0^t (\sigma_1 X_s + \sigma_2) \, dB_s, \quad t \in [0, T], \] (3.8)

for constants \( c_i \) and \( \sigma_i, i = 1, 2 \).

The above conditions are satisfied (check them!) for

\[ a(t, x) = c_1 x + c_2 \quad \text{and} \quad b(t, x) = \sigma_1 x + \sigma_2. \] (3.9)

An Itô stochastic differential equation (3.8) with linear (in \( x \)) coefficient functions \( a(t, x) \) and \( b(t, x) \) is called a linear Itô stochastic differential equation. In Section 3.3 we will give the solution of the general linear stochastic differential equation. By virtue of the above theory, linear stochastic differential equations have a unique strong solution on every interval \([0, T]\), whatever the choice of the constants \( c_i \) and \( \sigma_i \).

§ 3.2.2. Solving Itô Stochastic Differential Equations

by Itô Lemma

Example 3.2.4
The linear Itô stochastic differential equation with multiplicative noise:

\[ X_t = X_0 + c \int_0^t X_s \, ds + \sigma \int_0^t X_s \, dB_s, \quad t \in [0, T] \]

(\( c \) and \( \sigma \) are given constants)

has unique solution:

\[ X_t = X_0 \exp \left\{ (c - \frac{1}{2} \sigma^2) t + \sigma B_t \right\}, \quad t \in [0, T]. \]

This solution is a geometric Brownian Motion.

(See page 1, Lecture 10)
Use Itō lemma to find a solution to (*).

Suppose $X_t = f(t, B_t)$ would be a solution to (*). The apply Itō lemma we have

$$X_t = X_0 + \int_0^t \left[ f_1(s, B_s) + \frac{1}{2} f_{22}(s, B_s) \right] ds + \int_0^t f_2(s, B_s) dB_s$$

If we identify the integrand of (*I) with (**), we have

$$\begin{cases}
  c f(t, x) = f_1(t, x) + \frac{1}{2} f_{22}(t, x) \\
  \sigma f(t, x) = f_2(t, x)
\end{cases}$$

From (***) $\Rightarrow$ $\sigma^2 f(t, x) = f_{22}(t, x)$

Plug into (**) $\Rightarrow$ $c f(t, x) = f_1(t, x) + \frac{\sigma^2}{2} f(t, x)$
So the equation is
\[
\begin{align*}
(C - \frac{\sigma^2}{2}) f(t,x) &= f_1(t,x) \\
\sigma f(t,x) &= f_2(t,x)
\end{align*}
\]

Try to write \( f(t,x) = g(t) h(x) \). Then
\[
\begin{align*}
(c - \frac{\sigma^2}{2}) g(t) &= g'(t) \\
\sigma h(t) &= h'(t)
\end{align*}
\]

Solution of \( g \) and \( h \) can be found by separation of variables.
\[
\begin{align*}
g(t) &= g(0) e^{(c - 0.5\sigma^2)t} \\
h(x) &= h(0) e^{\sigma x}
\end{align*}
\]

Thus
\[
f(t,x) = g(0) h(0) \exp \left\{ (c - 0.5\sigma^2) t + \sigma x \right\}
\]

\[X_0 = f(0,0) = g(0) h(0)\]

\[X_t = f(t, X_0) = X_0 \exp \left\{ (c - 0.5\sigma^2) t + \sigma B_t \right\}, \quad t \in [0, T] \]

**Example 3.25** Consider the linear stochastic equation
\[
X_t = X_0 + c \int_0^t X_s \, ds + \sigma \int_0^t dB_s, \quad t \in [0, T]
\]

(Langevin equation (1908))
Equation (\ref{eq:1}) is related to time series analysis. Rewrite (\ref{eq:1}) as

$$dX_t = cX_t \, dt + \sigma \, dB_t$$

If $dt = \pm 1$, then the above becomes

$$X_{t+1} - X_t = cX_t + \sigma (B_{t+1} - B_t)$$

or

$$X_{t+1} = \phi X_t + Z_t$$

which is an autoregressive process of order 1.

To solve (\ref{eq:1}), let $Y_t = e^{-ct}X_t$. If $X_t$ is a solution to (\ref{eq:1}), then $Y_t$ is an Itô process. $Y_t = f(t, X_t)$ where $f(t, x) = e^{-ct}x$. Apply Itô lemma, extension 2, we have

$$Y_t - Y_0 = \int_0^t \left[ f_1(s, X_s) + A_s^{(1)} f_2(s, X_s) + \frac{1}{2} \left[A_s^{(2)} f_{22}(s, X_s) \right] \right] ds$$

$$+ \int_0^t A_s^{(2)} f_s(s, X_s) dB_s$$

But $A_s^{(1)} = cX_s$, $A_s^{(2)} = \sigma$, $f_1(t, x) = -cX(t, x)$, $f_2(t, x) = e^{-ct}$

$A_s^{(2)} f_s(s, X_s) = 0$. So (\ref{eq:1}) becomes

$$Y_t - Y_0 = \int_0^t [-cf(t, X_s) + cX_s e^{-cs}] \, ds + \int_0^t [B e^{-cs}] \, dB_s = \int_0^t [B e^{-cs}] \, dB_s$$
This concludes that if \( X_t \) is a solution to (31), then
\[
Y_t - Y_0 = e^{-ct} X_t - X_0 = \sigma \int_0^t e^{-cs} dB_s
\]
The solution to (31) is
\[
X_t = e^{ct} X_0 + \sigma e^{ct} \int_0^t e^{-cs} dB_s
\]
This is the so-called Ornstein-Uhlenbeck process.

Figure 3.2.6 Five sample paths of the Ornstein-Uhlenbeck process (3.17).
Left: \( X_0 = 1, c = 0.1, \sigma = 1 \). Right: \( X_0 = 10, c = -1, \sigma = 1 \).

This is an example that the solution to stochastic differential equation is given by a simple integral of Brownian Motion.

In order to verify that the process \( X \), given by (3.17), is actually a solution to (3.16), apply the Itô lemma (2.30) to the process \( X_t = u(t, Z_t) \), where
\[
Z_t = \int_0^t e^{-cs} dB_s \quad \text{and} \quad u(t, z) = e^{ct} X_0 + \sigma e^{ct} z.
\]
Moreover, since the Langevin equation is a linear Itô stochastic differential equation, we may conclude from Example 3.2.2 that \( X \) is the unique strong solution to (3.16).
We verify that the Ornstein–Uhlenbeck process is a Gaussian process. Assume for simplicity that $X_0 = 0$. Recall from the definition of the Itô stochastic integral that
\[ \int_0^t e^{-cs} dB_s \]
is the mean square limit of approximating Riemann–Stieltjes sums
\[ S_n = \sum_{i=1}^n e^{-c t_{i-1}} (B_{t_i} - B_{t_{i-1}}) \]
for partitions $\tau_n = (t_i)$ of $[0, t]$ with $\text{mesh}(\tau_n) \to 0$. The latter sum has a normal distribution with mean zero and variance
\[ \sum_{i=1}^n e^{-2c t_{i-1}} (t_i - t_{i-1}) . \] (3.18)
(Check this!) Notice that (3.18) is a Riemann sum approximation to the integral
\[ \int_0^t e^{-2cs} ds = \frac{1}{2c} \left( 1 - e^{-2ct} \right) . \]
Since mean square convergence implies convergence in distribution (see Appendix A1) we may conclude that the mean square limit $X_t$ of the normally distributed Riemann–Stieltjes sums $S_n$ is normally distributed. This follows from the argument given in Example A1.1 on p. 186. We have for $X_0 = 0$:
\[ EX_t = 0 \quad \text{and} \quad \text{var}(X_t) = \frac{\sigma^2}{2c} \left( e^{2ct} - 1 \right) . \]

Using the same Riemann–Stieltjes sum approach, you can also calculate the covariance function of an Ornstein–Uhlenbeck process with $X_0 = 0$:
\[ \text{cov}(X_s, X_t) = \frac{\sigma^2}{2c} \left( e^{c(t+s)} - e^{c(t-s)} \right) , \quad s < t . \] (3.19)
Since $X$ is a mean-zero Gaussian process, this covariance function is characteristic for the Ornstein–Uhlenbeck process.
Figure 3.2.7 Two paths of the two-dimensional process \((B_t^{(1)}, B_t^{(2)}), t \in [0, T]\), where \(B^{(1)}\) and \(B^{(2)}\) are two independent Brownian motions. See Example 3.2.8.

Example 3.2.8 (A stochastic differential equation with two independent driving Brownian motions)

Let \(B^{(i)} = (B_t^{(i)}, t \geq 0)\) be two independent Brownian motions and \(\sigma_i, i = 1, 2,\) real numbers. See Figure 3.2.7 for an illustration of the two-dimensional process \((B_t^{(1)}, B_t^{(2)})\).

Define the process
\[
\tilde{B}_t = (\sigma_1^2 + \sigma_2^2)^{-1/2} \left( \sigma_1 B_t^{(1)} + \sigma_2 B_t^{(2)} \right).
\]

Using the independence of \(B^{(1)}\) and \(B^{(2)}\), it is not difficult to see that
\[
E\tilde{B}_t = 0 \quad \text{and} \quad \text{cov}(\tilde{B}_t, \tilde{B}_s) = \min(s, t),
\]

i.e. \(\tilde{B}\) has exactly the same expectation and covariance functions as standard Brownian motion; see p. 35. Hence \(\tilde{B}\) is a Brownian motion.

Now consider the integral equation
\[
X_t = X_0 + c \int_0^t X_s \, ds + \sigma_1 \int_0^t X_s \, dB_s^{(1)} + \sigma_2 \int_0^t X_s \, dB_s^{(2)},
\]
for constants \(c\) and \(\sigma_i\).
We interpret the latter equation as

\[
X_t - X_0 = c \int_0^t X_s \, ds + \int_0^t X_s \, d\left[ \sigma_1 B_s^{(1)} + \sigma_2 B_s^{(2)} \right]
\]

\[= c \int_0^t X_s \, ds + (\sigma_1^2 + \sigma_2^2)^{1/2} \int_0^t X_s \, d\tilde{B}_s, \tag{3.20}\]

which is an Itô stochastic differential equation with driving Brownian motion \(\tilde{B}\). From Example 3.2.4 we can read off the solution:

\[
X_t = X_0 e^{\left[ c - 0.5 (\sigma_1^2 + \sigma_2^2)\right] t + (\sigma_1^2 + \sigma_2^2)^{1/2} \tilde{B}_t}
\]

\[= X_0 e^{\left[ c - 0.5 (\sigma_1^2 + \sigma_2^2)\right] t + [\sigma_1 B_t^{(1)} + \sigma_2 B_t^{(2)}]}.
\]

In the above definition (3.20) of the stochastic integral we were quite lucky because \(X\) appears as a multiplier in both integrals. Following similar patterns as for the definition of the Itô stochastic integral, it is also possible to introduce the stochastic integral

\[\int_0^t A_s^{(1)} \, dB_s^{(1)} + \int_0^t A_s^{(2)} \, dB_s^{(2)}\]

for more general processes \(A^{(i)}\). Moreover, the Brownian motions \(B^{(i)}\) can be dependent, and it is also possible to consider more than two driving Brownian motions. \(\Box\)