ASYMPTOTIC MINIMAX PROPERTIES OF L-ESTIMATORS OF SCALE

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Summary

We ask whether or not the efficient L-estimator of scale corresponding to the least informative distribution in \( \epsilon \)-contamination and Kolmogorov neighbourhoods of certain distributions possesses the saddlepoint property. This is of interest since the saddlepoint property implies the minimax property - that the supremum of the relative asymptotic variance of an L-estimator is minimized by the efficient estimator corresponding to that member of the distributional class with minimum Fisher information for scale. Our findings are negative in all cases investigated.

Key words: L-estimator of scale, minimum Fisher information for scale, relative asymptotic variance, \( \epsilon \)-contamination neighbourhood, Kolmogorov neighbourhood.

1. Introduction

Consider the L-estimate of scale defined as \( S(F_n) \), where \( F_n \) is the empirical distribution function based on a sample \( X_1, \ldots, X_n \sim F \), and the functional \( S(F) \) is defined by

\[
(S(F))^q = \int_0^1 (F^{-1}(t))^q dM(t)
\]

(1)

where \( M \) is a signed measure on \((0, 1)\) with a density \( m \) and \( q \) is an integer > 0. The case \( q = 1 \) includes scale estimators based on linear combinations of the ordered
sample, such as the interquartile range, semi-interquartile range. See Serfling (1980, pg. 263-267) for more examples. The case $q = 2$ includes various trimmed and Winsorized forms of the sample standard deviation.

Under appropriate regularity conditions, $\sqrt{n}(S(F_n) - S(F))$ is asymptotically normally distributed with zero mean and variance

$$V(m, F) = E_F[IC^2(x, F, m)]$$

where

$$IC(x, F, m) = \frac{1}{S(F)^{q-1}} \left[ \int_0^1 (s - 1) \frac{(F^{-1}(s))^{q-1}}{f(F^{-1}(s))} m(s) ds + \int_0^{F(x)} (F^{-1}(s))^{q-1} \frac{m(s)}{f(F^{-1}(s))} ds \right],$$

the influence curve of $S$. See Huber (1981 p.110) and Welsh and Morrison (1990) for details.

Suppose $F$ is only approximately known in the sense that $F$ is known to lie within a certain convex class $\mathcal{F}$ of distributions.

For $F = F_{(1)} \in \mathcal{F}$, define $F_{(\sigma)}$ by

$$F_{(\sigma)}(x) = F\left( \frac{x}{\sigma} \right), \quad \sigma > 0.$$  

Define the Fisher information for scale of a distribution $F$ on the real line by

$$I(F; 1) = \sup_{-\infty < x < \infty} \frac{\int_{-\infty}^{\infty} x \chi(x) dF(x)^2}{\int_{-\infty}^{\infty} \chi^2(x) dF(x)},$$

where the sup is taken over all continuously differentiable functions $\chi$ with compact support, satisfying $\int_{-\infty}^{\infty} \chi^2(x) dF(x) > 0$.

Note that if we put $F_{(\sigma)}(x) = F\left( \frac{x}{\sigma} \right)$ and $I(F; \sigma) = I(F_{(\sigma)}, 1)$, then we have $I(F_{(\sigma)}, 1) = \frac{1}{\sigma^2} I(F; 1)$.

Suppose that $F_0$ minimizes $I(F; 1)$ in $\mathcal{F}$ and let $m_0$ be the corresponding efficient choice of L-estimator of $\sigma$ if $X_1, ..., X_n \sim F_{0,(\sigma)}$. Define $S_0(F)$ by (1), with $m = m_0$ and define the relative asymptotic variance by

$$R(m, F) = \frac{V(m, F)}{S^2(F)}$$
In this paper, we address the question of whether or not
\[ R(m_0, F) \leq R(m_0, F_0) \leq R(m, F_0) \]  \hspace{1cm} (2)
holds, for all \( F \in \mathcal{F} \) for \( \mathcal{F} \) being equal to the \( \epsilon \)-contamination neighbourhood model
\[ \mathcal{G}_\epsilon(G) = \{ F \mid F = (1 - \epsilon)G + \epsilon H, \ H \text{ arbitrary} \}, \]
or the Kolmogorov neighbourhood model
\[ \mathcal{K}_\epsilon(G) = \{ F \mid \sup_{-\infty < x < \infty} | F(x) - G(x) | \leq \epsilon \} \]
with \( \epsilon \) and \( G \) known and \( G \) satisfying certain mild regularity conditions. By Cramér-Rao Inequality, the right hand inequality in (2) always holds.

If (2) is true, we say that the saddlepoint property holds. And hence we have
\[ \sup_{\mathcal{F}} R(m_0, F) = \inf_m \sup_{\mathcal{F}} R(m, F), \]  \hspace{1cm} (3)
so that minimax property holds: the maximum (over \( \mathcal{F} \)) value of the relative asymptotic variance is minimized by \( m_0 \). The quantity \( R(m, F) \) was first proposed by Daniel (1920) as a measure of accuracy of a scale estimator. Bickel and Lehmann (1976) named it the standardized asymptotic variance.

In the past two decades, the problem of checking whether or not various robust estimators possess the saddlepoint property has received a fairly large amount of attention. For the location case, see Huber (1964, 1981), Jaeckel (1971), Collins (1983), Collins and Wiens (1985, 1989), Sacks and Ylvisaker (1972, 1982), Wiens (1986, 1990). For the scale case, Huber (1981) numerically verified that for \( \epsilon \leq .04 \), the saddlepoint property holds for \( M \)-estimate of scale in \( \epsilon \)-contamination normal neighbourhood model, \( \mathcal{G}_\epsilon(\Phi) \) where \( \Phi \) is the standard normal distribution function. Wiens and Wu (1990) worked also on \( M \)-estimates of scale and showed that the saddlepoint property fails in \( \mathcal{G}_\epsilon(G) \) for sufficiently large \( \epsilon \) and \( \mathcal{K}_\epsilon(G) \) where \( G \) satisfies certain assumptions. With the aid of numerical calculation, they also show that the saddlepoint property fails in \( \mathcal{G}_\epsilon(\Phi) \) for \( .0997 \leq \epsilon \leq .2051 \).

In this paper, we show that the saddlepoint property (2) fails in:

1. \( \mathcal{G}_\epsilon(G) \) for large \( \epsilon \) (section 3.1).
2. \( \mathcal{G}_\epsilon(G) \) for moderate \( \epsilon \) (section 3.2).
3. \( \mathcal{K}_\epsilon(G) \), all \( \epsilon \) (section 3.3).
2. Some preliminaries

First we obtain $m_0$. By the Cramèr-Rao Inequality, $V(m, F) \geq \frac{1}{I(F, 1)}$ and equality holds if and only if

$$IC(x, F, m) = \frac{\chi(x)}{I(F, 1)}$$

where $\chi(x) = -x^T(x) - 1$, (Huber (1981, p. 68), Hampel et al (1986, p. 86)).

Thus $V(m_0, F_0) = \frac{1}{I(F_0, 1)}$ implies $\frac{d}{dx}IC(x, F_0, m_0) = \frac{\chi'_0(x)}{I(F_0, 1)}$. With $m_0(F_0(x)) = (\frac{d}{dx}IC(x, F_0, m_0))(\frac{S_0(F_0)}{x})^{-q-1}$, we have

$$m_0(F_0(x)) = \frac{\chi'_0(x)}{I(F_0, 1)}(\frac{S_0(F_0)}{x})^{-q-1}$$

That is

$$m_0(w) = \frac{\chi'_0(F_0^{-1}(w))(F_0^{-1}(w))^{-q+1}}{I(F_0, 1)}$$

where in the above, we have used the fact that $S_0(F_0) = 1$ (See (7) below).

Now put

$$\phi(t) = \frac{1}{R(m_0, F_t)}$$

where $F_t = (1 - t)F_0 + tF_1$, and $0 < t < 1$.

By a Taylor series expansion of $\phi(t)$ at the point zero, we have

$$\phi(t) = \phi(0) + t\phi'(0) + o(t) \quad (4)$$

If we can find an $F_1$ in $\mathcal{F}$ such that $\phi'(0) < 0$, then we are through since $\phi(t) < \phi(0)$ for sufficiently small $t$.

**Theorem 1.**

$$\phi'(0) = \int_{-\infty}^{\infty} [2q\chi_0(x) - \chi_0^2(x)]d(F_1 - F_0)(x)$$

$$-2\int_{-\infty}^{\infty} \chi_0(x) \int_{-\infty}^{x} \frac{\chi_0(y)}{f_0(y)}(F_1 - F_0)(y)dydF_0(x)$$

$$-2(1 - q)\int_{-\infty}^{\infty} \chi_0(x) \int_{-\infty}^{x} \frac{\chi_0'(y)}{yf_0(y)}(F_1 - F_0)(y)dydF_0(x) \quad (5)$$
where \( f_0 = F_0' \).

Proof: Firstly,
\[
\phi' (0) = \frac{2S_0(F_0)}{V(m_0, F_0)} \frac{d}{dt} S_0(F_t) \Big|_{t=0} - \left( \frac{S_0(F_0)}{V(m_0, F_0)} \right)^2 \frac{d}{dt} V(m_0, F_t) \Big|_{t=0}
\]

(6)

Note that
\[
S_0(F_0) = \left( \int_{-\infty}^{\infty} x^q m_0(F_0(x)) dF_0(x) \right)^{1/\alpha}
\]
\[
= \left( \int_{-\infty}^{\infty} \frac{x^q}{I(F_0, 1)} \left( \frac{S_0(F_0)}{x} \right)^{q^{-1}} dF_0(x) \right)^{1/\alpha}
\]
\[
= \left( S_0(F_0) \right)^{q^{-1}} \left( \int_{-\infty}^{\infty} \frac{x^q \chi_0''(x) dF_0(x)}{I(F_0, 1)} \right)^{1/\alpha}
\]
\[
= \frac{1}{I(F_0, 1)} \int_{-\infty}^{\infty} x \chi_0''(x) dF_0(x)
\]
\[
= \frac{1}{I(F_0, 1)} \int_{-\infty}^{\infty} \chi_0^2(x) dF_0(x)
\]

(7)

upon an integration by parts. Consequently, \( S_0(F_0) = 1 \) or equivalently,
\[
\int_0^1 (F_0^{-1}(w))^q m_0(w) dw = 1.
\]

Thus
\[
\frac{d}{dt} S_0(F_t) \Big|_{t=0} = \frac{d}{dt} \left[ \int_0^1 (F_t^{-1}(w))^q m_0(w) dw \right]^{1/\alpha} \Big|_{t=0}
\]
\[
= \frac{1}{q} \int_0^1 (F_t^{-1}(w))^{q-1} \frac{d}{dt} \int_0^1 (F_t^{-1}(w))^q m_0(w) dw \Big|_{t=0}
\]
\[
= \frac{1}{q} \int_0^1 \frac{d}{dt} (F_t^{-1}(w))^q \Big|_{t=0} m_0(w) dw
\]
\[
= \int_0^1 (F_t^{-1}(w))^{q-1} \frac{w - F_1(F_t^{-1}(w))}{f_0(F_t^{-1}(w))} m_0(w) dw
\]
\[
= \int_{-\infty}^{\infty} x^{q-1} \frac{(F_0(x) - F_1(x))}{f_0(x)} m_0(F_0(x)) dF_0(x)
\]
\[
= \int_{-\infty}^{\infty} x^{q-1} \chi_0(x) \left( \frac{F_0(x) - F_1(x)}{f_0(x)} \right)^{q^{-1}} dx
\]
\[
= \frac{1}{I(F_0, 1)} \int_{-\infty}^{\infty} \chi_0(x) [F_0(x) - F_1(x)] dx
\]
\[
= \frac{1}{I(F_0, 1)} \int_{-\infty}^{\infty} \chi_0(x) d(F_1 - F_0)(x)
\]

(8)
upon an integration by parts. On the other hand,

\[
\frac{d}{dt} V(m_0, F_t) \mid_{t=0} = \int_{-\infty}^{\infty} 2IC(x, F_0, m_0) \frac{d}{dt} IC(x, F_t, m_0) \mid_{t=0} dF_0(x) \\
+ \int_{-\infty}^{\infty} IC^2(x, F_0, m_0) d(F_1 - F_0)(x) \\
= 2 \int_{-\infty}^{\infty} \chi_0(x) \frac{d}{dt} IC(x, F_t, m_0) \mid_{t=0} dF_0(x) \\
+ \int_{-\infty}^{\infty} \chi_0^2(x) \frac{d}{dt^2} IC(F_0, F_t, m_0) \mid_{t=0} dF_0(x) \\
+ \int_{-\infty}^{\infty} \chi_0^2(x) \frac{d}{dt^2} IC(F_0, F_t, m_0) \mid_{t=0} dF_0(x)
\]  

(9)

Now

\[
\frac{d}{dt} IC(x, F_t, m_0) \mid_{t=0} = \left[ \int_0^1 (s - 1) \left( \frac{F_t^{-1}(s)}{f_0(F_0^{-1}(s))} \right)^{q-1} m_0(s) ds \\
+ \int_0^F_0(x) \left( \frac{F_t^{-1}(s)}{f_0(F_0^{-1}(s))} \right)^{q-1} m_0(s) ds \frac{d}{dt} \int_0^1 (s - 1) \left( \frac{F_t^{-1}(s)}{f_0(F_0^{-1}(s))} \right)^{q-1} m_0(s) ds \mid_{t=0} \\
+ \frac{1}{(S_0(F_0))^{q-1}} \int_0^1 \left( \frac{F_t^{-1}(s)}{f_0(F_0^{-1}(s))} \right)^{q-1} m_0(s) ds \mid_{t=0} \right]
\]

(10)

Since

\[
\frac{d}{dt} \frac{1}{S_0(F_t)^{q-1}} \mid_{t=0} = (-q + 1) \left( S_0(F_0) \right)^{-q} \frac{d}{dt} S_0(F_t) \mid_{t=0} \\
= \frac{(-q + 1)}{I(F_0, 1)} \int_{-\infty}^{\infty} \chi_0(x) d(F_1 - F_0)(x)
\]

(11)

\[
\int_{-\infty}^{\infty} \chi_0(x) \frac{d}{dt} IC(x, F_t, m_0) \mid_{t=0} dF_0(x) \\
= \int_{-\infty}^{\infty} \chi_0(x) dF_0(x) \left[ \int_0^1 (s - 1) \left( \frac{F_t^{-1}(s)}{f_0(F_0^{-1}(s))} \right)^{q-1} m_0(s) ds \frac{d}{dt} \left( S_0(F_t)^{q-1} \right) \mid_{t=0} \\
+ \frac{d}{dt} \int_0^1 (s - 1) \left( \frac{F_t^{-1}(s)}{f_0(F_0^{-1}(s))} \right)^{q-1} m_0(s) ds \mid_{t=0} \\
+ \frac{(-q + 1)}{I(F_0, 1)} \int_{-\infty}^{\infty} \chi_0(x) \int_0^{F_t(x)} \left( \frac{F_t^{-1}(s)}{f_0(F_0^{-1}(s))} \right)^{q-1} m_0(s) ds dF_0(x) \int_{-\infty}^{\infty} \chi_0(x) d(F_1 - F_0)(x) \\
+ \int_{-\infty}^{\infty} \chi_0(x) \mid \frac{d}{dt} \int_0^{F_t(x)} \left( \frac{F_t^{-1}(s)}{f_0(F_0^{-1}(s))} \right)^{q-1} m_0(s) ds \mid_{t=0} dF_0(x)
\]

6
\[
\begin{align*}
&= \frac{(-q+1)}{I(F_0, 1)} \int_{-\infty}^{\infty} \chi_0(x) \int_{-\infty}^{x} y^{q-1} m_0(F_0(y)) dy dF_0(x) \int_{-\infty}^{\infty} \chi_0(x) d(F_1 - F_0)(x) \\
&+ \int_{-\infty}^{\infty} \chi_0(x) \int_{-\infty}^{x} y^{q-1} \frac{d}{dt} m_0(F_t(y)) dy \bigg|_{t=0} dF_0(x)
\end{align*}
\]

as \( \int_{-\infty}^{\infty} \chi_0(x) dF_0(x) = 0. \) Moreover,

\[
\frac{d}{dt} m_0(F_t(y)) \bigg|_{t=0} = \frac{\chi_0''(y)(F_1 - F_0)(y)y^{-q+1}}{I(F_0, 1)f_0(y)} + \frac{(-q+1)\chi_0'(y)(F_1 - F_0)(y)y^{-q}}{I(F_0, 1)f_0(y)}.
\]

Putting (6) to (13) together, the result follows. □.

3. Saddlepoint properties

3.1. \( \mathcal{G}_\epsilon(G) \), large \( \epsilon \) case:

Assume that \( G \) is a distribution having a twice differentiable density such that \( \xi(x) = -x^{\frac{q-1}{q}} - 1 \) is increasing on \((0, \infty)\) and decreasing on \((-\infty, 0)\). Wu (1990) notes that these are only mild conditions on \( G \) which include many important distributions. Normal, logistic, Student’s t and Laplace are typical ones.

Under these conditions, Wu (1990) showed that there exists an \( \epsilon_s(G) \) such that for \( \epsilon > \epsilon_s(G) \), \( I(F, 1) \) is minimized over \( \mathcal{G}_\epsilon(G) \) by that \( F \) for which

\[
-x \frac{F_0'(x)}{f_0} - 1 = \chi_0(x) = \begin{cases} 
\lambda, & x \in (-\infty, m_1) \\
\xi(x), & x \in [m_1, m_2] \\
-\lambda, & x \in (m_2, m_3] \\
\xi(x), & x \in (m_3, m_4] \\
\lambda, & x \in (m_4, \infty) 
\end{cases}
\]

where \(-\infty < m_1 < m_2 < 0 < m_3 < m_4 < \infty\), \( \lambda > 0 \) and the constants \( \lambda, m_1, m_2, m_3, m_4 \) are determined by the side conditions that \( \chi_0(x) \) be continuous at \( m_1, m_2, m_3, m_4 \) and that \( \int_{-\infty}^{\infty} f_0(x) dx = 1. \)

**Theorem 2.** For any given \( \mathcal{G}_\epsilon(G) \) with \( \epsilon \geq \epsilon^*(G) \) (and therefore \( (F_0, \chi_0) \)), there exists a subset \( \mathcal{G}_\epsilon^1(G) \) of \( \mathcal{G}_\epsilon(G) \) such that for all \( F_1 \in \mathcal{G}_\epsilon^1(G) \),

\[ \phi'(0) < 0 \]

**Proof:** Consider the following subset of \( \mathcal{G}_\epsilon(G) \):

\[ \mathcal{G}_\epsilon^1(G) = \{ F_1 \in \mathcal{G}_\epsilon(G) \mid F_1 = (1 - \epsilon)G + \epsilon H_1, I(F_1, 1) < \infty \} \]
where

\[(H_0 - H_1)(m_1) = (H_0 - H_1)(m_2) = \delta_1 \geq 0,\]
\[(H_1 - H_0)(m_3) = (H_1 - H_0)(m_4) = \delta_2 \geq 0,\]

and

\[\delta_1 + \delta_2 > 0,\]

where \(H_0 = \frac{F_0 - (1-\epsilon)(G)}{\epsilon}\). Then we have

\[\int_{-\infty}^{\infty} \chi_0(x) d(F_1 - F_0)(x)\]
\[= (\int_{-\infty}^{m_1} + \int_{m_1}^{m_2} + \int_{m_2}^{m_3} + \int_{m_3}^{\infty}) \chi_0(x) d(F_1 - F_0)(x)\]
\[= (\lambda)(-\delta_1) + (-\lambda)(\delta_1 + \delta_2) + (\lambda)(-\delta_2)\]
\[= -2\lambda(\delta_1 + \delta_2)\]  \hspace{1cm} (15)

By the same technique, we can show that

\[\int_{-\infty}^{\infty} \chi_0^2(x) d(F_1 - F_0)(x) = 0\]  \hspace{1cm} (16)

Thus

\[\int_{-\infty}^{\infty} [2q\chi_0(x) - \chi_0^2(x)] d(F_1 - F_0) = -4q\lambda(\delta_1 + \delta_2) < 0.\]  \hspace{1cm} (17)

Obviously

\[\chi_0''(x)(F_1 - F_0)(x) = 0 ; \chi_0'(x)(F_1 - F_0)(x) = 0\]

for all \(x \in (-\infty, \infty)\). By Theorem 1,

\[\phi'(0) = -4q\lambda(\delta_1 + \delta_2) < 0.\] \hspace{1cm} □

3.2. \(G_\epsilon(G)\), moderate \(\epsilon\) case:

Under the same conditions on \(G\) as above, Wu (1990) shows that for \(\epsilon < \epsilon_4(G)\), \(I(F, 1)\) is minimized over \(G_\epsilon(G)\) by that \(F_0\) for which

\[-x \frac{f_0'}{f_0}(x) - 1 = \chi_0(x) = \begin{cases} \xi(x), & x \in (m_1, m_4) \\ \lambda, & x \in (m_1, m_4)^c \end{cases}\]  \hspace{1cm} (18)

where \(-\infty < m_1 < 0 < m_4 < \infty, \lambda > 0\) and the constants \(\lambda, m_1, m_4\) are determined by the side conditions that \(\chi_0(x)\) be continuous at \(m_1, m_4\) and that \(\int_{-\infty}^{\infty} f_0(x) dx = 1\).
Theorem 3. There exists an $\varepsilon^{**}(G) \geq 0$ such that for any given $G_{\varepsilon}(G)$ with $\varepsilon^{**} \leq \varepsilon \leq \varepsilon^*(G)$ (and therefore $(F_0, \chi_0)$), there exists a subset $G_{\varepsilon}^2(G)$ of $G_{\varepsilon}(G)$ such that for all $F_1 \in G_{\varepsilon}^2(G)$,

$$
\phi'(0) < 0.
$$

Proof: Consider the following subset of $G_{\varepsilon}(G)$:

$$
G_{\varepsilon}^2(G) = \{F_1 \in G_{\varepsilon}(G) \mid F_1 = (1 - \varepsilon)G + \varepsilon H, I(F_1, 1) < \infty\}
$$

where $(H_0 - H_1)(x) = \delta_3 > 0$ on $(m_1, 0)$ and $(H_0 - H_1)(x) = 0$ on $(0, \infty)$. Then

$$
\int_{-\infty}^{\infty} \chi_0(x) d(F_1 - F_0)(x) = \int_{-\infty}^{m_1} \lambda d(F_1 - F_0)(x) + \delta_3 \chi_0(0)
$$

$$
= -(\lambda + 1)\delta_3.
$$

(19)

as $\chi_0(0) = -1$. Also

$$
\int_{-\infty}^{\infty} \chi_0^2(x) d(F_1 - F_0)(x) = \int_{-\infty}^{m_1} \lambda^2 d(F_1 - F_0)(x) + \delta_3 \chi_0^2(0)
$$

$$
= \delta_3 (1 - \lambda^2)
$$

(20)

Thus By Theorem 1,

$$
\phi'(0) = -2q(\lambda + 1)\delta_3 - \delta_3 (1 - \lambda^2) = -(\lambda + 1)\delta_3 (2q + 1 - \lambda).
$$

Thus $\phi'(0) < 0$ if $(2q + 1 - \lambda) > 0$. As in Wu (1990), $\lambda$ is an decreasing function of $\varepsilon$, it follows that there exists an $\varepsilon^{**}(G) \geq 0$ such that whenever $\varepsilon^*(G) \geq \varepsilon \geq \varepsilon^{**}(G)$, we have $\phi'(0) < 0$. □

3.3. $K_{\varepsilon}(G)$, all $\varepsilon$:

Under certain mild assumptions, the form of $(\chi_0, F_0)$ was obtained by Wiens and Wu (1991). These assumptions are satisfied by normal, logistic, Student’s t, Laplace, etc.

There are several forms of $(\chi_0, F_0)$ depending on the value of $\varepsilon$. For precise listing of assumptions and the forms of $(\chi_0, F_0)$, the reader is referred to Wiens and Wu (1991) or Wu (1990). In each form, $(\chi_0, F_0)$ has the following properties:

a) $F_0$ is symmetric and there exist constants $0 < a \leq b < c \leq d < \infty$ such that

$$
| (F_0 - G)(x) | < \varepsilon, \quad x \in [0, a) \cup (b, c) \cup (d, \infty);
$$

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\[ | (F_0 - G)(x) | = \epsilon, \quad x \in [a, b] \cup [c, d]; \]
b) \( \chi_0 \) is symmetric, increasing in \( |x| \), being constant on each of \([0, a), [d, \infty)\) and equal to
\[
\lambda \tan \left( \frac{\lambda}{2} \log x + \omega \right)
\]
on \((b, c)\) where \( \lambda \) is a positive constant, \( \omega \in \mathbb{R} \).

Note that
\[
\frac{d^2}{dx^2} \lambda \tan \left( \frac{\lambda}{2} \log x + \omega \right) = \frac{d}{dx} \frac{\lambda^2}{2x} \sec^2 \left( \frac{\lambda}{2} \log x + \omega \right)
\]
\[
= -\frac{\lambda^2}{2x^2} \sec^2 \left( \frac{\lambda}{2} \log x + \omega \right) (1 - \chi_0(x))
\]
\[
= -\frac{\chi'_0(x)(1 - \chi_0(x))}{x}
\]
(21)

Lemma 1. For any given \( K_e(G) \) (and therefore \( (F_0, \chi_0) \)), the inequality
\[
-2 \int_{-\infty}^{\infty} \chi_0(x) \int_{-\infty}^{x} \left[ (1 - q) \frac{\chi_0'(y)}{y f_0(y)} + \frac{\chi_0(y)}{f_0(y)} \right] (F_1 - F_0)(y) dy f_0(x) < 0
\]
(22)
is satisfied by an appropriate choice of \( F_1 \).

Proof: Suppose there exists an interval \((k_1, k_2) \subseteq (b, c) \cup (-c, -b)\) such that \( F_1 \neq F_0 \) on \((k_1, k_2)\), \( F_1 = F_0 \) on \((k_1, k_2)^c\). Then the l.h.s. of (22) becomes

\[
-2 \int_{k_1}^{k_2} \chi_0(x) \int_{-\infty}^{\min(k_2, x)} \frac{\chi_0'(y)}{y f_0(y)} [\chi_0(y) - q] (F_1 - F_0)(y) dy f_0(x)
\]

Note that

A1). \( \frac{\chi'_0(y)}{y f_0(y)} \geq 0 \)

for all \( y \in \mathbb{R} - \{0\} \);
A2). If the product of the terms \( \chi_0(x), [(\chi_0(y) - q] \) and \( (F_1 - F_0)(y) \) is greater than zero on \((k_1, k_2)\), then the inequality in (22) is satisfied.

We will show that A2) can always be satisfied by showing that such interval \((k_1, k_2)\) always exists. Consider the product

\[
\chi_0(x) [\chi_0(y) - q] \quad \text{for } x \in (-\infty, \infty), \ y \in (-\infty, x)
\]

10
As $\chi_0(x)$ is monotone increasing and continuous on $(0, \infty)$, we can find an interval $(k_1, k_2) \in (b, c) \cup (-c, -b)$ such that one of the following holds:

B1) $\chi_0(x)|\chi_0(y) - q| > 0$, for $x, y \in (k_1, k_2)$.

B2) $\chi_0(x)|\chi_0(y) - q| < 0$, for $x, y \in (k_1, k_2)$.

Since $\chi_0(x)$ is symmetric, we can also conclude that for $x, y \in (-k_2, -k_1)$, either B1) or B2) holds.

If B1) holds, we choose $F_1 = F^*$ such that

$$
(F^* - F_0)(x) \begin{cases}
= 0 & x \in (k_1, k_2) \\
> 0 & x \in (k_1, k_2)
\end{cases}
$$

(24)

If B2) holds, we choose $F_1 = F^{**}$ such that

$$
(F^{**} - F_0)(x) \begin{cases}
= 0 & x \in (k_1, k_2) \\
< 0 & x \in (k_1, k_2)
\end{cases}
$$

(25)

Obviously, we can restrict our choice of $F^*$ or $F^{**}$ to lie within the Kolmogorov neighbourhood $K_e(G)$ and A2) is satisfied by the choice of $F^*$, or $F^{**}$, depending on whether B1) or B2) holds on $(k_1, k_2)$. Therefore the lemma is proved. □

**Lemma 2** For any given $K_e(G)$ (and therefore $(F_0, \chi_0)$), the inequality

$$
\int_{-\infty}^{\infty} [2q\chi_0(x) - \chi_0^2(x)]d(F_1 - F_0)(x) < 0
$$

(26)

is satisfied for an appropriate choice of $F_1$.

Proof: We again restrict $F_1 \neq F_0$ only on an interval $(k_3, k_4) \subseteq (b, c) \cup (-c, -b)$, and $F_1 = F_0$ otherwise. The choice of $(k_3, k_4)$ will be specified later. First consider the function

$$
A(x) = 2q\chi_0(x) - \chi_0^2(x).
$$

(27)

Note that

$$
\frac{dA(x)}{d\chi_0(x)} = 2(q - \chi_0(x)) \begin{cases}
> 0, & \text{if } \chi_0(x) < q \\
< 0, & \text{if } \chi_0(x) > q
\end{cases}
$$

(28)

Since $\chi_0(x)$ is increasing on $(0, \infty)$, $A'(x)$ changes sign at most once on $(0, \infty)$. Similarly, $A'(x)$ changes sign at most once on $(-\infty, 0)$ as $\chi_0(x)$ is symmetric. We now pick the interval $(k_3, k_4)$ such that one, but not both, of the following is true: C1). $A(x)$ is increasing for $x \in (k_3, k_4)$.

C2). $A(x)$ is decreasing for $x \in (k_3, k_4)$.
If C1) holds in \((k_3, k_4)\), then for some \(k_5, k_6\) where \(k_3 < k_5 < k_6 < k_4\), \(F_1\) is specified by
\[
d(F_1 - F_0)(x) = \begin{cases} 
0 & x \in (k_3, k_4) \cup (k_5, k_6) \\
> 0 & x \in (k_3, k_5) \\
< 0 & x \in (k_6, k_4)
\end{cases}
\] (29)

Then
\[
\int_{-\infty}^{\infty} A(x) d(F_1 - F_0)(x) = \left( \int_{k_3}^{k_5} + \int_{k_6}^{k_4} \right) A(x) d(F_1 - F_0)(x) \\
\leq A(k_5) \int_{k_3}^{k_5} d(F_1 - F_0)(x) + A(k_6) \int_{k_6}^{k_4} d(F_1 - F_0)(x) \\
= (A(k_5) - A(k_6)) \int_{k_3}^{k_5} d(F_1 - F_0)(x) \\
< 0.
\] (30)

If however, C2) holds on \((k_3, k_4)\), then for some \(k_7, k_8\) where \(k_3 < k_7 < k_8 < k_4\), we choose \(F_1\) such that
\[
d(F_1 - F_0)(x) = \begin{cases} 
0 & x \in (k_3, k_4) \cup (k_7, k_8) \\
< 0 & x \in (k_3, k_7) \\
> 0 & x \in (k_8, k_4)
\end{cases}
\] (31)

Using the same argument as in (30), (26) also holds. Since the interval \((k_3, k_4)\) always exists, the lemma is proved. \(\Box\)

**Theorem 4.** For any given \(K_c(G)\) (and therefore \((F_0, \chi_0)\)), there exists a family \(\mathcal{F}^*\) of \(F_1\)'s, such that
\[
\phi'(0) < 0
\] (32)

Proof: By Lemma 1, there always exists a family of distributions with members defined by either (24) or (25), depending on whichever is appropriate, will guarantee (22) be satisfied. So we only need to show that within this family, there exists a subset \(\mathcal{F}^*\) such that (26) holds for members of \(\mathcal{F}^*\).

From Lemma 1, we know that there always exists an interval \((k_1, k_2) \subset (b, c) \cup (-c, -b)\) such that one of B1) or B2) is true. Let us consider the case that B1) holds in \((k_1, k_2)\). In this case, the family \(\mathcal{F}_1\) containing those distributions defined by (24) is appropriate for the inequality (22) to hold. Furthermore, when B1) holds in \((k_1, k_2)\), it must also hold in \((-k_2, -k_1)\) because of symmetry. Therefore, the family \(\mathcal{F}_2\) containing those distributions defined as in (24), but over \((-k_2, -k_1)\) is also appropriate for (22) to hold.
Now within \((k_1, k_2)\), we can always find an interval \((k_3, k_4)\) such that one of \(C1\) or \(C2\) is true. If \(C1\) holds in \((k_3, k_4)\), then the class of distributions \(\mathcal{F}^*\) satisfying (29) over \((k_3, k_4)\) will make (26) true and \(\mathcal{F}^*\) is a subset of \(\mathcal{F}_1\), so in this case (32) follows.

If on the other hand, \(C2\) holds in \((k_3, k_4)\), then \(C1\) must hold in the region \((-k_4, -k_3)\) because of symmetry. Therefore, the class of distributions \(\mathcal{F}^*\) satisfying (29) over \((-k_4, -k_3)\) makes (26) true and \(\mathcal{F}^*\) is a subset of \(\mathcal{F}_1\), so in this case, (32) follows.

Now we consider the case that \(B2\) holds in \((k_1, k_2)\). In that case, the family \(\mathcal{F}_3\) with members defined by (25) makes the inequality (22) holds. Also \(B2\) will hold in \((-k_2, -k_1)\). Thus the family \(\mathcal{F}_4\) defined by (25) but on \((-k_2, -k_1)\) is also appropriate for (22) to hold.

When \(B1\) is true, we can find \((k_5, k_6) \subseteq (k_1, k_2)\) such that one of \(C1\) or \(C2\) is true. If \(C2\) holds, defining \(\mathcal{F}^*\) by those distributions defined by (31) makes (26) hold and \(\mathcal{F}^*\) is a subset of \(\mathcal{F}_3\) and so (32) follows. Finally, if \(C1\) holds in \((k_5, k_6)\), \(C2\) must hold in \((-k_6, -k_5)\) and defining \(\mathcal{F}^*\) by those distributions defined by (31) over \((-k_4, -k_3)\) will make (26) valid. Furthermore, \(\mathcal{F}^*\) is a subset of \(\mathcal{F}_4\) and (32) also follows in this final case.

Obviously, members of \(\mathcal{F}^*\) can be chosen to lie within \(\mathcal{K}_c(G)\). □

References