Chapter 4

The Itô lemma and the Black-Scholes formula

Denote $f(t, S)$ as a function depending on the time and the asset price $S$. Since the asset value $S$ follows GBM, the total differential of the function $f(t, S)$ should be obtained through the Itô lemma, which is a mathematical method for tackling functions depending on stochastic variables.

**The Itô Lemma**

Suppose $S$ follows the dynamics:

$$dS = a(t, S)dt + b(t, S)dW; \quad dW \sim N(0, dt).$$

Then, the total differential of $f(t, S)$ is

$$df(t, S) = \left[ \frac{\partial f}{\partial t} + a(t, S)\frac{\partial f}{\partial S} + \frac{1}{2} b^2(t, S)\frac{\partial^2 f}{\partial S^2} \right] dt + b(t, S)\frac{\partial f}{\partial S} dW. \quad (4.1)$$

**Informal Justification:**

Consider the Taylor series expansion for the function:

$$df(t, S) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{1}{2} \left[ \frac{\partial^2 f}{\partial t^2} (dt)^2 + 2 \frac{\partial^2 f}{\partial t \partial S} (dt)(dS) + \frac{\partial^2 f}{\partial S^2} (dS)^2 \right] + \cdots.$$ 

and

$$
(dt)(dS) = a(t, S)(dt)^2 + b(t, S)(dW)(dt)
$$

$$
(dS)^2 = [a(t, S)dt + b(t, S)dW]^2 = a^2(dt)^2 + 2ab(dt)(dW) + b^2(dW)^2.
$$

The differential form of a function collects all the terms up to order $dt$. That means we regard all terms of the form $(dt)^n$ as zero if $n > 1$. One tricky
thing is that the term \( dW \) which can be viewed as
\[
dW = \epsilon \sqrt{dt} \quad \epsilon \sim N(0, 1).
\]
Therefore, we have
\[
(dt)^2 \approx 0 \quad \text{because} \quad 2 > 1
\]
\[
(dt)(dW) = \epsilon(dt)^{1.5} \approx 0 \quad \text{because} \quad 1.5 > 1
\]
and
\[
(dW)^2 = \epsilon^2 dt \approx dt \quad \text{because} \quad E(\epsilon^2) = 1.
\]
As a result, we have
\[
(dt)^2 = 0, \quad (dt)(dS) = 0 \quad \text{and} \quad (dS)^2 = b(t, S)^2 dt.
\]
Substituting the above into the Taylor series expansion and neglecting higher order terms yield the Itô lemma.

### 4.1 Application of the Itô lemma

Suppose we want to investigate the rate of return of an asset. Then, we may start by two possible approaches: the percentage rate of return, \( R_t = (S_t - S_{t-1})/S_{t-1} \) or the log return, \( R_t = \ln(S_t/S_{t-1}) \). Is there any difference between them? Using Itô’s lemma, we see that their drift are different but the volatilities are the same.

Consider the model of percentage rate of return in continuous time that
\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t. \tag{4.2}
\]

If the standard differentiation technique is employed, we find that
\[
\frac{d \ln S}{dS} = \frac{1}{S} \Rightarrow d \ln S = \frac{dS}{S} = \mu dt + \sigma dW_t.
\]

In other words, two models are identical. In contrast, if we denote \( f(t, S) = \ln S \), then the Itô lemma tells us that
\[
d \ln S = \left( \mu S \frac{\partial \ln S}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \ln S}{\partial S^2} \right) dt + \sigma S \frac{\partial \ln S}{\partial S} dW
\]
\[
- \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dW. \tag{4.3}
\]

The Itô lemma says that standard differentiation is not useful for a function with random input. An additional term is required to describe the contribution of randomness. In the particular example above, this additional term is \(-\frac{\sigma^2}{2} dt\).
4.1.1 Expected asset price

The Itô lemma can also be used to compute expectation. We illustrate ideas from calculating the expected asset price, $E(S_T)$. Suppose the asset price follows the Black-Scholes dynamics of (5.13). Then, the dynamics for the log-return becomes (5.14). This enables us to take integration to both side and to yield

$$
\int_0^T d \ln S_t = \int_0^T \left( \mu - \frac{\sigma^2}{2} \right) dt + \int_0^T \sigma dW_t
$$

$$
\ln S_T - \ln S = \left( \mu - \frac{\sigma^2}{2} \right) T + \sigma (W_T - W_0)
$$

$$
S_T = S e^{\left( \mu - \frac{\sigma^2}{2} \right) T + \sigma \varepsilon W_T}
$$

$$
S_T = S e^{\left( \mu - \frac{\sigma^2}{2} \right) T + \sigma \varepsilon \sqrt{T}} \epsilon \sim N(0, 1).
$$

(4.4)

Having this kind of expression, the expectation can be obtained through evaluating

$$
E(S_T) = \int_{-\infty}^{\infty} S e^{\left( \mu - \frac{\sigma^2}{2} \right) T + \sigma \varepsilon \sqrt{T}} \phi(\varepsilon) d\varepsilon,
$$

where

$$
\phi(\varepsilon) = \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2},
$$

(4.5)

is the probability density function for a standard normal random variable. Let us do it in detail.

$$
\int_{-\infty}^{\infty} S e^{\left( \mu - \frac{\sigma^2}{2} \right) T + \sigma \varepsilon \sqrt{T}} \phi(\varepsilon) d\varepsilon = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S e^{\left( \mu - \frac{\sigma^2}{2} \right) T + \sigma \varepsilon \sqrt{T}} e^{-\varepsilon^2/2} d\varepsilon
$$

$$
= Se^{\mu T} \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(\varepsilon - \sigma \sqrt{T})^2/2} d\varepsilon
$$

$$
= Se^{\mu T}
$$

(4.6)

If $\mu = r$, i.e. in the risk-neutral world, then we have

$$
E^Q(S_T) = Se^{rT},
$$

which is the futures price for the asset.

4.1.2 Improve the speed of simulating option prices

For derivatives pricing, we consider the risk neutral world where every security has expected rate of return equal to the risk-free rate. In other words,
the risk neutral world can be constructed by replacing $\mu$ (for every security) in (4.2) by the risk-free interest rate, namely $r$.

With the help of the Itô lemma, option prices can be valued via (4.3) instead of discretizing (4.2). This improves the accuracy as well as the speed for estimating option prices. In the risk neutral world, the discrete version of (4.3) reads,

$$\ln S_{t-\Delta t} - \ln S_t = \left( r - \sigma^2/2 \right) \Delta t + \sigma (W_{t-\Delta t} - W_t).$$

As the terminal asset price is concerned for valuing European options, we take summation to the above equation. It arrives at a version of (4.4) with $\mu$ is replaced by $r$:

$$S_T = S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma \epsilon \sqrt{T}} \epsilon \sim \mathcal{N}(0,1).$$  \hspace{1cm} (4.7)

The MC simulation can be performed as follows:
Step 1: Generate $n$ iid $\epsilon \sim \mathcal{N}(0,1)$
Step 2: Compute $n$ terminal asset prices, $S_T^{(j)}$, $j = 1, 2, \cdots, n$, by (4.7).
Step 3: Estimate the call price as

$$\text{Call} \simeq e^{-rT} \frac{1}{n} \sum_{j=1}^{n} \max(S_T^{(j)} - K, 0).$$

![Figure 4.1: Comparison of (4.2) and (4.4).](image)

### 4.2 The Black-Scholes formula

In this section, we would study the Black-Scholes model with the help of the Itô lemma. We would like to answer the following three questions:

1. Why are derivatives securities independent to the drift of the underlying assets?
2. What is the Black-Scholes equation?

3. What is the Black-Scholes formula?

### 4.2.1 The Black-Scholes Equation

We begin with considering a simple contingent claim having the contract function \( C(S(T)) \). Denote the derivatives price process \( \Pi(t; C) = F(t, S(t)) \).

We start by computing the price dynamics of the derivative asset, and apply the Itô formula to give us

\[
dF = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial S} dS + \frac{1}{2} \frac{\partial^2 F}{\partial S^2} (dS)^2
\]

\[
= \left\{ \frac{\partial F}{\partial t} + \alpha S \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} \right\} dt + \frac{\partial F}{\partial S} dW
\]

\[
= \alpha_F(t)F(t)dt + \sigma_F(t)F(t)dW(t)
\]

where the processes \( \alpha_F \) and \( \sigma_F \) are defined by

\[
\alpha_F = \frac{F_t + \alpha S F_S + \frac{1}{2} \sigma^2 S^2 F_{SS}}{F}, \quad (4.9)
\]

\[
\sigma_F = \frac{\sigma S F_S}{F} \quad (4.10)
\]

Let us now form a portfolio based on two assets: the underlying stock and the derivative asset. Denoting the relative portfolio by \((u_S, u_F)\) we obtain the following dynamics for the value \( V \) of the portfolio.

\[
dV = V \left\{ u_S \frac{dS}{S} + u_F \frac{dF}{F} \right\}
\]

\[
= V[u_S\alpha dt + \sigma dW] + u_F[\alpha_F dt + \sigma_F dW]
\]

\[
= V[u_S\alpha + u_F \alpha_F]dt + V[u_S\sigma + u_F \sigma_F]dW. \quad (4.11)
\]

The only restriction on the relative portfolio is that we must have

\[
u_S + u_F = 1,
\]

for all \( t \). Let us thus define the relative portfolio by the linear system of equations

\[
u_S + u_F = 1 \quad (4.12)
\]

\[
u_S\sigma + u_F \sigma_F = 0. \quad (4.13)
\]

Using this portfolio we see that by its definition the driving \( dW \)-term in the \( V \)-dynamics of (4.11) vanishes completely, leaving us with the equation

\[
dV = V[u_S\alpha + u_F \alpha_F]dt.
\]
Thus we have a locally riskless portfolio, and because of the requirement that the market is free of arbitrage, we must have the relation

$$u_S \alpha + u_F \alpha_F = r,$$  \hspace{1cm} (4.14)

meaning that any riskless portfolio should earn risk-free interest rate. Otherwise, arbitrage opportunity occurs. Solving (4.12)-(4.13), we have

$$u_S = \frac{\sigma_F}{\sigma_F - \sigma},$$

$$u_F = \frac{-\sigma}{\sigma_F - \sigma},$$

which, using (4.10), gives us the portfolio more explicitly as

$$u_S = \frac{SF_s}{SF_s - F},$$ \hspace{1cm} (4.15)

$$u_F = \frac{-F}{SF_s - F},$$ \hspace{1cm} (4.16)

Now, we substitute (4.9), (4.15) and (4.16) into the absence of arbitrage condition (4.14). Then, after some calculation, we obtain the equation

$$F_t + rSF_s + \frac{1}{2} \sigma^2 S^2 F_{SS} - rF = 0.$$

Furthermore, $$F(T, S(T)) - c(S(T)).$$

**Theorem 4.1 (Black-Scholes Equation)** The only price function of the form $$\Pi(t; C) = F(t, S)$$ which is consistent with the absence of arbitrage is when $$F$$ is the solution of the following terminal value problem in the domain $$[0, T] \times R^+$$.

$$\frac{\partial F}{\partial t} + rS \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - rF = 0,$$

$$F(T, S) = C(S).$$ \hspace{1cm} (4.17)

**Remarks:**

1. The Black-Scholes equation governs every European option (like call, put and others). For valuing different options, the user is required to specify the contract function $$C(S)$$ only.

2. It is seen that the governing equation is independent to the drift of the underlying asset. Instead, we observe the interest rate $$r$$ appearing in several places of the equation.

3. This equation is derived from the no arbitrage argument alone.
4.2.2 Deriving the Black-Scholes formulas

Forward value

We can derive pricing formulas for various derivatives from the Black-Scholes equation. For instance, we would like to demonstrate the pricing of a forward contract value. A forward value has the payoff of

\[ S_T - K, \]

where \( K \) is the delivery price. Since the payoff is a linear function of the underlying asset price, it is logical to guess that the forward value is a linear function of the underlying asset price as well. Therefore, let us try the guess of

\[ f(t, S) = a(t)S + b(t), \]

where we require that

\[ f(T, S) = a(T)S_T + b(T) = S_T - K \Rightarrow a(T) = 1 \text{ and } b(T) = -K. \]

Substitute the guess into the governing equation, we have

\[ [a'(t)S + b'(t)] + ra(t)S - r[a(t)S + b(t)] = 0. \]

It follows that

\[ a'(t) = 0 \text{ and } b'(t) = rb(t) \Rightarrow a(t) \equiv 1 \text{ and } b(t) = -Ke^{-r(T-t)}. \]

This gives the (Black-Scholes) forward value as \( S - Ke^{-r(T-t)} \), since the formula is derived from the Black-Scholes equation.

European call option

The second example is devoted to the pricing formula for European call options. The terminal payoff happens to be

\[ \max(S_T - K, 0). \]

We denote the call option price by \( F(t, S) \). That means the no arbitrage price of the option will be obtained if the user puts the current time \( t \) and the current asset price \( S \) into the pricing function \( F(t, S) \). Since the function depends on the underlying asset price, a stochastic variable, it is logical to consider the total differential of the function. By the Itô lemma, the differential form reads

\[ dF = \left( \frac{\partial F}{\partial t} + rS \frac{\partial F}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 F}{\partial S^2} \right) dt + \sigma S \frac{\partial F}{\partial S} dW \]
if the underlying asset price evolves as a Geometric Brownian motion in the risk-neutral world, i.e.

\[
\frac{dS}{S} = r dt + \sigma dW.
\]

By the Black-Scholes equation, we know that the coefficient of \( dt \) is identical to the term \( rF \), see (4.17). The total differential for the pricing function is simplified as

\[
dF = rF dt + \sigma S \frac{\partial F}{\partial S} dW \Rightarrow dF - rF dt = \sigma S \frac{\partial F}{\partial S} dW.
\]

Now, we guess \( F(t, S) = e^{rt} f(t, S) \). This choice enables us to combine the two terms in the left-hand side into a single differentiation by product rule of differentiation. It is easy to check that

\[
d \left[ e^{rt} f(t, S) \right] = e^{rt} df(t, S) + r e^{rt} f(t, S) dt = e^{rt} df(t, S) + rF dt,
\]

which implies that

\[
e^{rt} df(t, S) = \sigma S \frac{\partial F}{\partial S} dW \Rightarrow df(t, S) = \sigma e^{-rt} S \frac{\partial F}{\partial S} dW.
\]

The above differential form has an equivalent integration form which is

\[
f(T, S_T) - f(t, S) = \sigma \int_t^T e^{-\tau t} S \frac{\partial F}{\partial S} dW.
\]

The right-hand side of the above equation is viewed as a sum of Wiener processes so that it has an expected value of zero. After taking expectation to both side, under the risk neutral measure, we arrive at

\[
E^Q(f(T, S_T)) = E^Q(f(t, S)) = f(t, S).
\]

Using the relationship between \( f(t, S) \) and \( F(t, S) \), we know that

\[
F(t, S) = e^{rt} E^Q \left[ e^{-rT} F(T, S_T) \right] = e^{-r(T-t)} E^Q[\max(S_T - K, 0)]. \tag{4.19}
\]

Remarks:

1. The Black-Scholes equation is useful if the underlying asset price evolves as in the risk-neutral world. Otherwise, the total differential form for the function cannot be simplified. It is the advantage of risk-neutral pricing.

2. The risk-neutral probability \( Q \) refers to the process of the underlying asset price with drift being the risk-free interest rate.
To derive the Black-Scholes formula for a call option, we employ (4.4) and re-write (4.19) as

\[ e^{-r(T-t)} E^Q[\max(S e^{(r-\sigma^2/2)(T-t)} e^{\sigma \sqrt{T-t}}, 0), \epsilon] \sim \mathcal{N}(0,1). \]

Let us concentrate on the expectation. We have learned that

\[ E^Q[\max(S e^{(r-\sigma^2/2)(T-t)} e^{\sigma \sqrt{T-t}}, 0)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \max(S e^{(r-\sigma^2/2)(T-t)} e^{\sigma \sqrt{T-t}}, 0) e^{-\epsilon^2/2} d\epsilon. \]

The integrand has a non-zero value only when

\[ S e^{(r-\sigma^2/2)(T-t)} e^{\sigma \sqrt{T-t}} - K > 0 \Rightarrow \epsilon > -\frac{\ln \frac{S}{K} + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}. \]

We denote the right-hand side by \(-d_2\). Hence, the integration becomes

\[ \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} (S e^{(r-\sigma^2/2)(T-t)} e^{\sigma \sqrt{T-t}} - K) e^{-\epsilon^2/2} d\epsilon \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} S e^{(r-\sigma^2/2)(T-t)} e^{\sigma \sqrt{T-t}} e^{-\epsilon^2/2} d\epsilon - \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} K e^{-\epsilon^2/2} d\epsilon \]

\[ = S e^{r(T-t)} N(d_2 + \sigma \sqrt{T-t}) - K N(d_2) \]

After discounting the expected payoff, we have

\[ F(t, S) = S N(d_1) - K e^{-r(T-t)} N(d_2), \]

which is the celebrated Black-Scholes formula for a call option.

**European put option**

Of course, one can repeat the above step to value the European put option. A more clever way to obtain the put option price uses the put-call parity relation, which is a model-independent result.

Recall that the put-call parity:

\[ C_E + K e^{-r(T-t)} = P_E + S. \]

Therefore, the put option premium is

\[ P_E = K e^{-r(T-t)} [1 - N(d_2)] S [1 - N(d_1)] - K e^{-r(T-t)} N(d_2) SN(d_1). \]