Chapter 6
Simulation and Value at Risk

In this chapter
- Definition of VaR (parametric approach)
- Historical VaR approach
  --> basic concepts of VaR
- Measuring VaR for option

Risk Measures
Def: Each risk measure is a single real number representing different risk levels.
Example 1: variance of the portfolio return, this measure tells how the variable our return is, but doesn't tell the lose in monetary terms.
Example 2: max possible loss, this measure is not informative, since the max possible loss is usually the whole of the portfolio.

Backgrounds
- Till Guildiman, head of global research at JP Morgan in late 1980s, creator of the term “value at risk”.
- The G-30 provided a venue for discussing best risk management practices.
- In July 1993, the term “value at risk” was published in the G-30 report.

Definition of VaR
Value at Risk (VaR) summaries the worse loss over a target horizon with a given level of confidence.

Two parameters
- The horizon period, which might be daily, weekly, monthly, quarterly, yearly or whatever.
- the level of confidence, which might be 90%, 95%, 99% or any other probability we choose.
VaR describes the quantile of the projected distribution of gains and losses over the target horizon.

If \( c \) is the selected confidence level, VaR corresponds to the \( 1 - c = \alpha \) lower-tail level. For instance, with a 95% confidence level, VaR should be such that it exceeds 5 percent of the total number of observations in the distribution.

Numerical example

The ABC bank 1995 daily returns

<table>
<thead>
<tr>
<th>Daily Return</th>
<th>Number of Days</th>
</tr>
</thead>
<tbody>
<tr>
<td>-25</td>
<td>2</td>
</tr>
<tr>
<td>-21</td>
<td>1</td>
</tr>
<tr>
<td>-17</td>
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<tr>
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<td>15</td>
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<td>15</td>
<td>19</td>
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<tr>
<td>19</td>
<td>23</td>
</tr>
</tbody>
</table>

Mean = $5m

5% occurrence

Given horizon period as daily and 95% confidence, which is equivalent to regard the worst 5% daily revenues as unexpected loss scenarios due to risk.

We can also define VaR in terms of absolute dollar loss (absolute VaR), or in terms of loss relative to the mean revenue (relative VaR).

Absolute VaR

The former is simply the maximum amount we expect to lose with a given level of confidence, measured from our current level of wealth.

For ABC over 1995, the value is $10 million, the 13th (=5% \times 260) worst loss out of 260 trading days. This $10m is known as the absolute VaR for ABC over 1995.

Relative VaR

The alternative VaR is measured relative to the mean expected revenue over the period. This relative VaR figure is therefore obtained by adding the daily mean revenues to the absolute VaR. For ABC bank over 1995, the value is $10m + $5m = $15m.

Absolute VaR = -REV*
Relative VaR = Absolute VaR + Mean Revenue

where REV* is the cut-off revenues conditioning on the confidence level and horizon we choose.
VaR can also be represented in terms of the rate of return (the return) on the portfolio. The return can be regarded as the revenue divided by the initial value of the portfolio. If we denote the return by \( R \), the initial portfolio value by \( W \), and the revenue by \( REV \).

We have
\[
REV = R \times W \implies REV^* = R^* W.
\]

This expression enables us to transform VaR figures in terms of revenues into VaR in terms of returns, which are often more convenient to deal with.

Abs.
\[
\text{VaR} = -R^* W \implies \text{Abs. VaR in return} = -R^*
\]

Rel.
\[
\text{VaR} = -R^* W + \mu W \implies \text{Rel. VaR in return} = -R^* + \mu.
\]

Advantages of historical approach

This method is relatively simple to implement if historical data have been collected in-house for daily marking-to-market. The same data can be stored for later reuse in estimating VaR.

Advantages of historical approach

This method deals directly with the choice of horizon for measuring VaR. For instance, to obtain a monthly VaR, the user could re-construct historical monthly returns over, say, 5 years.

Perhaps most important, it does not depend on the probability distribution we assume for the return.

Problems of historical approach

This historical approach requires a “sufficient history” of price changes. To obtain 1000 independent returns of a 1-day move, we require 4 years of continuous data. If monthly return is concerned, 1000 independent returns stand for more than 80 years data.

Only one sample path is used so that it generates difficulties for scenarios analysis.

Problems of historical approach

Historical approach will be very slow to incorporate structural breaks, which are handled more easily with an analytical methods such as RiskMetrics.

It may not be suitable for analyzing derivatives risk. Derivatives products can have a very short life, say 3 months, in the market. Therefore, one may want to make use of the underlying asset prices. However, incorporating the prices of underlying assets can be terribly difficult because no model is assumed to relate underlying asset price and the derivatives price. Otherwise, the distribution-free advantage of the method is broken.
The method puts the same weight on all observations, including old data points. It could produce a very large error in estimating VaR. For instance, a 99% daily VaR estimated over 100 days produces only one observation in the tail, which necessarily leads to an imprecise VaR measure. Thus very long sample paths are required to obtain meaningful figure. The dilemma is that this may involve observations that are not relevant.

The method becomes cumbersome for large portfolios with complicated structures. In practice, users adopt simplifications such as grouping interest rate payoffs into bands, which considerably increase the speed of computation. But if too many simplifications are carried out, the benefit of distribution-free valuation can be lost.

Parametric VaR: Normal assumption

The parametric VaR focuses on the random process that describes the behavior of the asset's (portfolio's) return. That is we make assumption about the probability density function (pdf) of return. One common approach considers the normal distribution for the return process. Does the return really follow normal?

Returns

Arithmetic return
\[ R_t^A = \frac{P_{t+1} - P_t}{P_t} = \delta P_t / P_t \geq -100\% \]
where \( P_t \) is the price of the asset at time \( t \).

This implies that the return does not follow a normal distribution since a normal random variable can take any real number.

Geometric return
\[ R_t^G = \ln[P_{t+1}/P_t] = \ln P_{t+1} - \ln P_t = \delta \ln P_t \]
It can be any real number. (good!)

The differential forms for arithmetic return and geometric return are the same.
\[ d\ln P_t / dP_t = 1/P_t \Rightarrow \delta \ln P_t \equiv \delta P_t / P_t \Rightarrow R_t^G \equiv R_t^A. \]

See Table 6.1 of lecture note

The first column shows a series of asset prices generated from a simulation program.
The arithmetic return and geometric return for the asset are computed in the second and the third column respectively. The bottom part of the table shows means and variances of each approach.
It is seen that the difference between the two means is significant but their variances are of similar values.

Moreover, the mean of the arithmetic return minus the half of the geometric return variance produces $0.0984 - 0.5 \times 0.1416 = 0.0276$, which is close to the mean of geometric return.

This verifies that mean of $R^G \equiv \text{mean of } R^A - \frac{1}{2} \text{ (variance)}$ and variance of $R^G \equiv \text{variance of } R^A$.

Ito lemma

- Recall:
  Suppose $\frac{dP}{P} = \mu dt + \sigma dZ$, $dZ \sim N(0, dt)$.

- Applying Ito lemma, we have
  $$d\ln P = (\mu - \sigma^2/2) dt + \sigma dZ.$$ 

Summary

- We know that the geometric return can be assumed to follow a normal distribution.
- If the mean and variance for the geometric return are found to be $\mu_G$ and $\sigma_G^2$, respectively. Then, we should aware that $\sigma_A = \sigma_G$ and $\mu_A = \mu_G + \sigma_G^2 / 2$

Normal VaR in return

Suppose we want to determine the absolute VaR $=-R^*$ with confidence level $c$ (or tail probability $\alpha = 1 - c$). Given the pdf for the return, we should be able to compute the cut-off return by

$$\Pr[R < R^*] = \alpha. \quad (6.8)$$

Normal VaR in return

Consider $R \sim N(\mu, \sigma^2)$. We can always transform $R$ into a standard normal random variable:

$$Z = \frac{R - \mu}{\sigma} \sim N(0,1)$$
Hence, (6.8) becomes
\[ \Pr\left( Z < \frac{R^* - \mu}{\sigma} \right) = N\left( \frac{R^* - \mu}{\sigma} \right) = \alpha \]
\[ \frac{R^* - \mu}{\sigma} = N^{-1}(\alpha) = z_\alpha \]

For instance, \( z_\alpha = -1.65 \) for \( \alpha = 5\% \) (95\% confidence) and
\[
\begin{array}{c}
\text{Area} = 5\% \\
-1.65 \\
0 \\
x
\end{array}
\]

\( z_\alpha = -2.33 \) for \( \alpha = 1\% \) (99\% confidence)
\[
\begin{array}{c}
\text{Area} = 1\% \\
-2.33 \\
0 \\
x
\end{array}
\]

This gives us the normal VaR measures:
Absolute VaR in return
\[ = -R^* = -\sigma z_\alpha \cdot \mu \]
Relative VaR in return
\[ = -R^* + \mu = -\sigma z_\alpha \]

As the variances computed from arithmetic returns and geometric returns are the same, two approaches produce the same relative VaR.

One should aware of the absolute VaR since the mean return appears in the expression.

It is possible that the geometric return is used for computation because of the normality assumption.
But, the user think that the arithmetic return or the VaR in revenue would be much more meaningful in practice. Then basically, we have two ways to deal with it.
1. If you would like to see the VaR in revenue only, you can apply the definition for the geometric return to retrieve the VaR in revenue and 
\[(R^g)^* = \ln(P_t^*/W) \Rightarrow P_t^* = W \exp[(R^g)^*] \] 
\[\Rightarrow \text{REV}^* = P_t^* - W.\]

2. For VaR in arithmetic return, consider the relationship of (6.7). We have
Absolute VaR in arithmetic return 
\[= -\sigma_z \cdot (\mu_C + \sigma^2/2)\]
Abs. VaR in revenue 
\[= \text{Abs. VaR in arithmetic return} \times W\]

**Holding period adjustment**

- If we are given information based on one particular holding period, but want VaR information based on a different holding period.
- Imagine we are still dealing with daily return, but are now interested in a longer horizon period of, say, 20 days (i.e. one business month).

- The VaR for the longer holding period is then:
\[\text{VaR}_{\text{monthly}} = -z_\alpha \sigma_{\text{daily}} W \sqrt{20}\]
\[= \sqrt{20} \text{ VaR}_{\text{daily}}\]

- In general, people present the normal (relative) VaR measure as
\[\text{VaR}_{\text{t}}^\alpha \text{ (relative)} = -z_\alpha \sigma_{\text{yearly}} \sqrt{T} \times W\]

- Provided that daily returns are independently distributed from one day to the next, the mean return over the 20-day period is:
\[\mu_{\text{monthly}} = 20\mu_{\text{daily}}\]
- Similarly, the variance of the 20-day return is:
\[\sigma^2_{\text{monthly}} = 20\sigma^2_{\text{daily}}\]
\[\Rightarrow \sigma_{\text{monthly}} = \sigma_{\text{daily}} \sqrt{20}\]

- The above formula indicates that the market risk of a position is increase with volatility, initial investment cost, the square root of the holding period and the users' risk averseness.
- Given the same portfolio, a more risk averse investor would like to have a larger confidence in the portfolio's performance.
- Hence, a smaller \(\alpha\) is chosen. A smaller \(\alpha\) gives a greater VaR. Do you think it matches with your understanding of market risk?
Does return follow normal?
- Checking the distribution
- Eyeball testing: Q-Q plot

Checking the distribution
- Kolmogorov-Smirnov Goodness of Fit:
  Let $F_n(x)$ to be the empirical distribution
  and $F_0(x)$ to be the hypothesized dist. The
  Kolmogorov-Smirnov statistic:
  $$D_n = \sup_x |F_n(x) - F_0(x)|.$$ 
  Test: $H_0: F(x) = F_0(x)$ vs $H_1: F(x) \neq F_0(x)$

Statistical Software (S-Plus) provides details for
the above test.

An example

Data vs normal  Data vs t(2)

The p-values of KS test:
0.7942 0.9610

Models for fitting fat tail
- Mixture Normal:
  $$R_t = \mu + \sigma_1 \epsilon_1 + \delta \sigma_2 \epsilon_2$$
  where $\epsilon_1$ and $\epsilon_2$ are iid $N(0,1)$, $\sigma_2 > \sigma_1$
  and $\delta = 0$ with probability $p$ and 1 with
  probability $1-p$ independent of $\epsilon_1$ and $\epsilon_2$.

Estimating: $\sigma_2$ and $p$.

Mixture of two normal
distribution

How to calculate VaR from a
proposed distribution?

- If we have strong statistical
  knowledge, we may obtain the VaR
  explicitly through hand-calculation.
- Usually, hand-calculation fails.
- Simulation is the unique approach.
Monte Carlo Simulation to VaR

MC method estimates VaR on the basis of simulation, that means to produce possible future events, derived from statistical models. It involves a number of steps.

- **Step 1:** Select an appropriate statistical model for the asset(s).
- **Step 2:** Construct fictitious price paths for the random variables involved.
- **Step 3:** Repeat Step 2 for enough times.
- **Step 4:** Read off the VaR from the proxy distribution.

Understanding MCS

Example: Suppose the geometric return of a stock follows normal:

\[
\ln(S_{t+1}) - \ln(S_t) = R = \mu + \sigma \epsilon_t,
\]

where \(\epsilon_t \sim N(0,1)\). By drawing 1000 \(\epsilon_t\)'s, we are able to obtain two information:

1. 1000 possible stock prices in the future.
2. The 51\(^{st}\) smallest return is the 95% VaR.

***The confidence interval for the VaR can also be obtained by repeating the above simulation for 100 times. The 95% confidence interval is the interval between the 3\(^{rd}\) smallest and the 3\(^{rd}\) largest VaRs.

Some extensions

The above idea can easily be extended to handle the following situations:

- \(\epsilon_t\) follows a t-distribution, which may be a better one to fit the data.
- A portfolio contains two assets. However, the covariance among assets should be considered during the simulation procedure.
- The prices of derivatives and fixed income securities, like options and bonds, can also be obtained.

Advantages of Simulations

- The implementation is simple.
- They are widely used in financial sectors.
- They can easily handle the price risks associated with non-linear positions. Eg. Exotic derivatives position.
- They can provide estimation error of VaR for any distributions.
- They can generate future scenarios as much as you want.
- They help to estimate parameters.

A heavy tail distribution.

- Generalized Error Distribution (GED):

\[ R_t = \mu_t + \sigma \epsilon_t \]

where \(\epsilon\) follows a GED with density function given by:

\[ f(\epsilon) = \frac{v \exp\left(-\frac{1}{2} \frac{\epsilon^2}{\lambda}\right)}{\lambda 2^{v/2} \Gamma(1/v)} \]

and

\[ \lambda = \left( \frac{2}{\Gamma(3/v)} \right)^{3/2} \frac{2^{1/2}}{\Gamma(1/v)} \]

It becomes normal when \(v = 2\).

Show an example with EXCEL here

Suppose \(p = 0.1, \mu = 0.05\),

\[ \sigma_1 = 0.1, \sigma_2 = 0.6, \]
The shape of GED