RMS 4001 Assignment 3 Suggested Solution

1. The S-Plus Program for the option:

```r
rm()
S0<-60   #set the initial stock price
dt<-1/260  #deltat
N<-5000  #number of Paths
T2<-131
vol<-0.25  #set the volatility
rc<-0.05  #set the interest rate
K<-50   #strike price
H<-45   #barrier level
#setup a matrix to store the stock prices
S<-matrix(rep(S0,T2*N),N,T2)
for (i in 1:(T2-1)){
z<-rnorm(N)
#updating by differential method
S[,i+1]<-S[,i]*(1+r*dt+vol*sqrt(dt)*z)
}

payoff<-pmax((S[,T2]-K),0)  #option payoff
for  (j in 1:N){
if (min(S[j,])<H)
  payoff[j]<-0
}
docprice<-exp(-r*0.5)*mean(payoff)
docprice

Answer:  11.72911
```

2(a). \( X_t = \sigma_1 (W_{t_1} - W_{t_2}) - \sigma_2 (W_{t_1} - W_{t_0}) \), \( t > t_2 > t_1 > t_0 \)

Let \( A = (W_{t_1} - W_{t_2}) \sim N(0, t-t_2) \) and \( B = (W_{t_1} - W_{t_0}) \sim N(0, t_1 - t_0) \)

The Brownian Motion, \( W \), has the property of independent increments: \( W \) is said to be have independent increments if for every choice of \( t_i \)

\( t_1 < t_2 < \cdots < t_n \) and \( n \geq 1 \), \( W_{t_1} - W_{t_2}, \cdots, W_{t_n} - W_{t_{n-1}} \) are independent random variables. (*)

By this property, \( A \) and \( B \) are independent normal random variables.

Since \( X_t = \sigma_1 A - \sigma_2 B \) is a linear combination of independent normal random variables, \( X_t \) follows normal distribution with mean = \( E(X_t) \) and variance = \( \text{var}(X_t) \).

\[
E(X_t) = E[\sigma_1 A - \sigma_2 B] \\
\Rightarrow E(X_t) = \sigma_1 E[A] - \sigma_2 E[B] \\
\Rightarrow E(X_t) = \sigma_1 (0) - \sigma_2 (0) = 0 \\
\text{var}(X_t) = \text{var}[\sigma_1 A - \sigma_2 B] \\
\Rightarrow \text{var}(X_t) = \sigma_1^2 \text{var}(A) + \sigma_2^2 \text{var}(B) - 2\sigma_1\sigma_2 \text{cov}(A,B) \\
\Rightarrow \text{var}(X_t) = \sigma_1^2 (t-t_2) + \sigma_2^2 (t_1-t_0) - 2\sigma_1\sigma_2 (0) \\
\Rightarrow \text{var}(X_t) = \sigma_1^2 (t-t_2) + \sigma_2^2 (t_1-t_0) \\
\therefore X_t \sim N(0, \sigma_1^2 (t-t_2) + \sigma_2^2 (t_1-t_0))
\]
2(b). \[ X_t = \sigma_1(W_t - W_{t_0}) - \sigma_2(W_{t_1} - W_{t_0}) \quad t > t_1 > t_2 > t_0 \]

\((W_t - W_{t_2})\) and \((W_t - W_{t_0})\) are not independent since there is overlap between the time interval \([t_2, t]\) and \([t_0, t_1]\).

So we decompose the \(X_t\) as follows:

\[ X_t = \sigma_1(W_t - W_{t_2}) - \sigma_2(W_{t_1} - W_{t_2}) \]

\[ \Rightarrow X_t = \sigma_1(W_t - W_{t_1} + W_{t_1} - W_{t_2}) - \sigma_2(W_{t_1} - W_{t_2} + W_{t_2} - W_{t_0}) \]

\[ \Rightarrow X_t = \sigma_1(W_t - W_{t_1}) + \sigma_1(W_{t_1} - W_{t_2}) - \sigma_2(W_{t_1} - W_{t_2}) + \sigma_2(W_{t_2} - W_{t_0}) \]

\[ \Rightarrow X_t = \sigma_1(W_t - W_{t_1}) + (\sigma_1 - \sigma_2)(W_{t_1} - W_{t_2}) + \sigma_2(W_{t_2} - W_{t_0}) \]

\[ \Rightarrow X_t = \sigma_1 C + (\sigma_1 - \sigma_2)D + \sigma_2 F \]

where \( C = (W_t - W_{t_1}) \sim N(0, t - t_1) \)

\( D = (W_{t_1} - W_{t_2}) \sim N(0, t_1 - t_2) \)

\( F = (W_{t_2} - W_{t_0}) \sim N(0, t_2 - t_0) \)

By the property of independent increments (*) and \( X_t \) is a linear combination of independent normal random variables, \( X_t \) follows normal distribution with mean = \( E(X_t) \) and variance = \( \text{var}(X_t) \).

\[
E(X_t) = \sigma_1 E(C) + (\sigma_1 - \sigma_2)D - \sigma_2 F
\]

\[
\Rightarrow E(X_t) = \sigma_1 E(C) + (\sigma_1 - \sigma_2)E(D) - \sigma_2 E(F)
\]

\[
\Rightarrow \text{var}(X_t) = \text{var}(\sigma_1 C + (\sigma_1 - \sigma_2)D - \sigma_2 F)
\]

\[
\Rightarrow \text{var}(X_t) = \sigma_1^2 \text{var}(C) + (\sigma_1 - \sigma_2)^2 \text{var}(D) - \sigma_2^2 \text{var}(F)
\]

\[
\Rightarrow \text{var}(X_t) = \sigma_1^2 (t - t_1) + (\sigma_1 - \sigma_2)^2 (t_1 - t_2) - \sigma_2^2 (t_2 - t_0)
\]

\[
\therefore X_t \sim N(0, \sigma_1^2 (t - t_1) + (\sigma_1 - \sigma_2)^2 (t_1 - t_2) - \sigma_2^2 (t_2 - t_0))
\]

2(c). \[ X_t = \sum_{j=1}^{n} f(W_{t_j}) (W_{t_j} - W_{t_{j-1}}) \]

The distribution of \( X_t \) cannot be obtained, since we cannot express it as a linear combination of independent normal random variable. However we can compute the expected value and variance of \( X_t \).

\[
E[X_t] = E[\sum_{j=1}^{n} f(W_{t_{j-1}})(W_{t_j} - W_{t_{j-1}})]
\]

\[
E[X_t] = E[f(W_{t_0})(W_t - W_{t_0}) + f(W_{t_1})(W_{t_2} - W_{t_1}) + \cdots + f(W_{t_{n-1}})(W_{t_n} - W_{t_{n-1}})]
\]

\[
E[X_t] = E[f(W_{t_0})(W_t - W_{t_0}) + E[f(W_{t_1})(W_{t_2} - W_{t_1})] + \cdots + E[f(W_{t_{n-1}})(W_{t_n} - W_{t_{n-1}})]
\]

By the property of independent increment (*), \( f(W_{t_{j-1}}) \) and \( (W_{t_j} - W_{t_{j-1}}) \) are independent. \( \Rightarrow \)

\[
E[X_t] = E[f(W_{t_0})]E[(W_t - W_{t_0})] + E[f(W_{t_1})]E[(W_{t_2} - W_{t_1})] + \cdots + E[f(W_{t_{n-1}})]E[(W_{t_n} - W_{t_{n-1}})]
\]

\[
E[X_t] = E[f(W_{t_0})](0) + E[f(W_{t_1})](0) + \cdots + E[f(W_{t_{n-1}})](0)
\]

\[
E[X_t] = 0
\]
The given text contains a variety of mathematical expressions and equations, which appear to be related to stochastic processes, possibly involving Brownian motion or similar random processes. The text is fragmented and contains various mathematical notations, such as variances, expectations, and integrals, alongside some natural text explaining the context of these mathematical expressions. The text is written in a formal mathematical style, typical of a textbook or research paper on stochastic calculus or related fields.
\[ g(t, X) = tX_t^2, \text{ by Ito Lemma:} \]

\[
\frac{\partial g}{\partial t} = X_t^2, \quad \frac{\partial g}{\partial X} = 2tX_t, \quad \frac{\partial^2 g}{\partial X^2} = 2t, \quad a(t, X) = \mu, \quad b(t, X) = \sigma
\]

\[ dg = \left[ X_t^2 + 2\mu t X_t + \sigma^2 t \right] dt + 2\sigma t X_t d\mathcal{W}_t
\]

3(b). \hspace{1cm} dX_t = \mu dt + \sigma d\mathcal{W}_t, d\mathcal{W}_t \sim N(0, dt) \hspace{1cm} (1)

\[ X_t = X_0 + \mu t + \int_0^t \sigma d\mathcal{W}_s \hspace{1cm} (2)
\]

\[ A_t = \frac{1}{t} \int_0^t X_s d\tau \]

By using integration by part:

\[ A_t = \frac{1}{t} \left[ [\tau X_\tau]_0^t - \int_0^t \tau dX_\tau \right] \]

\[ A_t = \frac{1}{t} t X_t - \frac{1}{t} \int_0^t \sigma \mu \tau d\tau - \frac{1}{t} \int_0^t \sigma \tau d\mathcal{W}_t \]

by substitute (1)

\[ A_t = X_0 + \mu t + \int_0^t \sigma d\mathcal{W}_s - \frac{\mu t}{2} - \frac{1}{t} \int_0^t \sigma \tau d\mathcal{W}_t \]

by substitute (2)

\[ E[A_t] = E[X_0 + \mu t + \int_0^t \sigma d\mathcal{W}_s - \frac{\mu t}{2} - \frac{1}{t} \int_0^t \sigma \tau d\mathcal{W}_t] \]

\[ E[A_t] = X_0 + \frac{\mu t}{2} \]

\[ \operatorname{var}[A_t] = \operatorname{var}[X_0 + \mu t + \int_0^t \sigma d\mathcal{W}_s - \frac{\mu t}{2} - \frac{1}{t} \int_0^t \sigma \tau d\mathcal{W}_t] \]

\[ \operatorname{var}[A_t] = \operatorname{var}[\int_0^t \sigma d\mathcal{W}_s - \frac{1}{t} \int_0^t \sigma \tau d\mathcal{W}_t] \]

\[ \operatorname{var}[A_t] = \frac{\sigma^2}{t^2} \operatorname{var}[\int_0^t (t - \tau) d\mathcal{W}_t] \]

\[ \operatorname{var}[A_t] = \frac{\sigma^2}{t^2} \operatorname{var}[\int_0^t \mathcal{W}_t d\tau] \]

By using (#) from 2(c)

\[ \operatorname{var}[A_t] = \frac{\sigma^2}{t^2} \int_0^t E[(t - \tau)^2] d\tau \]

\[ \operatorname{var}[A_t] = \frac{\sigma^2}{t^2} \int_0^t (t - \tau)^2 d\tau \]

\[ \operatorname{var}[A_t] = \frac{\sigma^2}{t^2} \left[ \frac{(t - \tau)^3}{3} \right]_0^t \]

\[ \operatorname{var}[A_t] = \frac{\sigma^2 t}{3} \]
3(c). \[ \text{rm}() \]
\begin{verbatim}
X0<-70    #initial value
mean<-0.5   #parameters for the dynamic
sigma<-0.4
dt<-0.01   #delta t
length<-1   #length of the Brownian Motion
n<-1/dt    #number of time steps
N<-1        #number of paths
X<-matrix(rep(X0,(n+1)*N),N,(n+1))
for (i in 1:n){
z<-rnorm(N)
X[,i+1]<-X[,i]+mean*dt+sigma*sqrt(dt)*z
}
A<-sum(X)/n
X[, (n+1)]
A
\end{verbatim}
Answer: \( X_1 = 70.3916, \ A_1 = 70.80447 \)

3(d). \[ \text{rm}() \]
\begin{verbatim}
X0<-70    #initial value
mean<-0.5   #parameters for the dynamic
sigma<-0.4
dt<-0.01   #delta t
length<-1   #length of the Brownian Motion
n<-1/dt    #number of time steps
N<-1000    #number of paths
X<-matrix(rep(X0,(n+1)*N),N,(n+1))
for (i in 1:n){
z<-rnorm(N)
X[,i+1]<-X[,i]+mean*dt+sigma*sqrt(dt)*z
}
A<-rowSums(X)/n
meanA<-mean(A)
varA<-var(A)
meanX<-mean(X[, (n+1)])
varX<-var(X[, (n+1)])
covAX<-var(A,X[, (n+1)])
meanA
varA
meanX
varX
covAX
\end{verbatim}
Answer: \( \text{mean}(A_1) = 70.25067, \ \text{var}(A_1) = 0.053774, \)
\( \text{mean}(X_1) = 70.51433, \ \text{var}(X_1) = 0.1644345, \)
\( \text{cov}(A_1, X_1) = 0.08108946 \)

3(e). \[ E[A_t] = X_0 + \frac{\mu^2}{2} = 70 + \frac{0.5 \times 1}{2} = 70.25, \quad \text{var}[A_t] = \frac{\sigma^2 t}{3} = \frac{0.4^2}{3} = 0.05333 \]
\[ E[X_t] = X_0 + \mu t = 70 + 0.5 = 70.5, \quad \text{var}[X_t] = \sigma^2 t = 0.4^2 = 0.16 \]
Since $A_t = X_t - \frac{\mu t}{2} - \frac{1}{2}\int_0^t \sigma \alpha dW_t$
\[\Rightarrow \text{var}(A_t - X_t) = \text{var}\left(-\frac{\mu t}{2} - \frac{1}{2}\int_0^t \sigma \alpha dW_t\right)\]
\[\Rightarrow \text{var}(A_t) + \text{var}(X_t) - 2\text{cov}(A_t, X_t) = \int_0^t \text{E}[(\frac{\sigma \alpha}{t})^2] d\tau\]
\[\Rightarrow \frac{\sigma^2 t}{3} + \sigma^2 t - 2\text{cov}(A_t, X_t) = \frac{\sigma^2 t}{3}\]
\[\Rightarrow \text{cov}(A_t, X_t) = \frac{\sigma^2 t}{2}\]
\[\text{cov}(A_t, X_t) = \frac{\sigma^2 t}{2} = \frac{0.4^2}{2} = 0.08\]

The results obtained from the simulation match with the theoretical values.

4(a).
\[\text{GM}_t = \left(\prod_{j=1}^{n} S_{j,t}\right)^{\frac{1}{n}}\]
\[\Rightarrow \text{GM}_t = \exp\left\{\ln\left(\prod_{j=1}^{n} S_{j,t}\right)^{\frac{1}{n}}\right\}\]
\[\Rightarrow \text{GM}_t = \exp\left\{-\ln\left(\prod_{j=1}^{n} S_{j,t}\right)\right\}\]
\[\Rightarrow \text{GM}_t = \exp\left\{-\frac{1}{n}\sum_{j=1}^{n} \ln S_{j,t}\right\}\]

We take limit $\Delta t \rightarrow 0$
\[\lim_{\Delta t \rightarrow 0} \text{GM}_t = \lim_{\Delta t \rightarrow 0} \exp\left\{-\frac{1}{n} \sum_{j=1}^{n} \ln S_{j,t}\right\}\]
\[\Rightarrow \lim_{\Delta t \rightarrow 0} \text{GM}_t = \exp\left\{-\frac{1}{n} \sum_{j=1}^{n} \ln S_{j,t}\right\}\]
\[\Rightarrow \lim_{\Delta t \rightarrow 0} \text{GM}_t = \exp\left\{-\int_0^t \ln S_{j,t} d\tau\right\}\]

4(b).
\rm()
\S0<-50 \quad \# initial value
\rate<-0.05 \quad \# parameters for the dynamic
\vol<-0.25
\dt<-1/260 \quad \# delta t
\length<-1 \quad \# length of the Brownian Motion
\n<-1/dt \quad \# number of time steps
\N<-5000 \quad \# number of paths
\S<-matrix(rep(S0,(n+1)*N),N,(n+1))

for (i in 1:n){
    \z<-rnorm(N)
    \S[,i+1]<-\S[,i]+(1+rate*dt+vol*sqrt(dt))*\z
}
\GM<-exp(rowSums(log(S))/(n+1))
\payoff<-pmax(GM-S[, (n+1)], 0)
\floatstrikeaoprice<-mean(payoff)*exp(-rate*length)

Answer: 2.116204
4(c).

```r
rm()
S0<-50    #initial value
rate<-0.05  #parameters for the dynamic
vol<-0.25
dt<-1/260   #delta t
length<-1  #length of the Brownian Motion
n<-1/dt   #number of time steps
N<-5000   #number of paths
K<-50    #strike price
S<-matrix(rep(S0,(n+1)*N),N,(n+1))

for (i in 1:n){
z<-rnorm(N)
S[,i+1]<-S[,i]*(1+rate*dt+vol*sqrt(dt)*z)
}

GM<-exp( rowSums( log(S) )/(n+1) )
payoff<-pmax(GM-K,0)
fixstrikegaoprice<-mean(payoff)*exp(-rate*length)
fixstrikegaoprice

Answer: 3.153471
```

4(d).

```r
rm()
S0<-50    #initial value
rate<-0.05  #parameters for the dynamic
vol<-0.25
dt<-1/260   #delta t
length<-1  #length of the Brownian Motion
n<-1/dt   #number of time steps
N<-5000   #number of paths
K<-50    #strike price
S<-matrix(rep(S0,(n+1)*N),N,(n+1))

for (i in 1:n){
z<-rnorm(N)
S[,i+1]<-S[,i]*(1+rate*dt+vol*sqrt(dt)*z)
}

AM<-rowSums(S)/(n+1)
payoff<-pmax(AM-K,0)
fixstrikeaaoprice<-mean(payoff)*exp(-rate*length)
fixstrikeaaoprice

Answer: 3.518227
```

4(e). The arithmetic average is larger than the geometric average. Consider the terminal payoff: \( \max(AM_T - K, 0) > \max(GM_T - K, 0) \). We expect the price for d) should be more expensive than c). The result of simulation matches with the expectation.