[Example: Option Pricing in FX Market]

Suppose that in the US dollar markets the current Sterling exchange rate is 1.5. Consider a European call option that offers the holder the right to buy 100 pounds for 150 US dollars at time T. The riskless borrowing borrowing rate in the UK is \( u \) and that in the US is \( r \). Assuming a single period binary model in which the exchange rate at the expiry time is either 1.65 or 1.45, find the fair price of this option.

\[
\begin{align*}
\text{risk neutral prob.:} \\
\text{In the US market} & \quad \mathbb{E}[1.5 e^{1.5 t} \, \text{P}] \\
& \quad \mathbb{E}[1.5 e^{1.5 t} \, (1 - \text{P})] \\
1.5 = e^{-rT} \left[ 1.5 e^{uT} \text{P} + 1.5 e^{uT} (1 - \text{P}) \right] \\
\Rightarrow \text{P} = 8.25 - 7.5 e^{(r-u)T} \\
\text{In the UK market} & \quad \mathbb{E}[1.5 e^{uT} \, \text{P}] \\
& \quad \mathbb{E}[1.5 e^{uT} \, (1 - \text{P})] \\
1 = e^{-rT} \left[ 1.5 e^{uT} \text{P} + 1.5 e^{uT} (1 - \text{P}) \right]
\end{align*}
\]

[Conclusion]

\[
\begin{align*}
\Rightarrow \text{P} &= 7.975 e^{(u-r)T} - 7.25 \\
\text{option price:} \\
\text{In the US market} & \quad V_0 = \left[ 15(1 - \text{P}) \right] e^{-rT} \\
& \quad = 112.5 e^{-uT} - 108.75 e^{-rT} \\
& \quad \text{(in dollars)} \\
\text{In the UK market} & \quad \text{V}_0 = \left[ \frac{15}{1.65} (1 - \text{P}) \right] e^{-uT} \times 1.5 \\
& \quad = 112.5 e^{-uT} - 108.75 e^{-rT} \\
& \quad \text{(in dollars)} \\
V_0 &= \text{V}_0
\end{align*}
\]

1. The risk-neutral probabilities calculated by a dollar trader and a Sterling trader are different.
2. The dollar cost at time zero of the option valued by either a dollar trader or a Sterling trader is the same.
[Exercise 1.13]

Suppose that $S_1, S_2$ have payoffs $(10, -10, 3)$ and $(9, -9, 2)$ in 3 possible scenarios at $T$.

- If $S^0 = (1,1)'$. Is there any arbitrage opportunity?
- If $S^0 = (1,0.8)'$. Is there any arbitrage opportunity?

For any asset $A$ with initial value $A^0 > 0$ and payoff $(A^1, \ldots, A^m)$, one way to quantify risk is by the criterion

$$ Risk = \max_{j=1, \ldots, m} \left| \frac{A^j}{A^0} \right|. $$

- Find the Risk for the portfolio $V_a = S_1$.
- Find the Risk for the portfolio $V_b = S_1 - S_2$.
- Can you construct a portfolio using $S_1$ and $S_2$ with minimum Risk?

\begin{align*}
(1) \quad S &= \begin{pmatrix} 10 \\ -10 \\ 0 \end{pmatrix} \\
S'q &= S_0 \Rightarrow \text{The state price vector does not exist} \Rightarrow \text{arbitrage}
\end{align*}

\begin{align*}
(2) \quad S'q &= S_0 \Rightarrow q = \left( \frac{1}{5}, \frac{1}{10}, \frac{1}{3} \right)'
\Rightarrow \text{arbitrage-free}.
\end{align*}

\begin{align*}
(3) \quad (A^1, A^2, A^3) &= (10, -10, 3) \\
A^0 &= 1
\Rightarrow Risk = \max \left( \left| \frac{10W_1 + 9W_2}{W_1 + 0.8W_2} \right|, \left| \frac{3W_1 + 3W_2}{W_1 + 0.8W_2} \right| \right)
\end{align*}

\begin{align*}
(4) \quad (A^1, A^2, A^3) &= (1, -1, 1) \\
A^0 &= 0.2 \\
\Rightarrow Risk &= \max \left( \left| \frac{10W_1 + 9W_2}{W_1 + 0.8W_2} \right|, \left| \frac{3W_1 + 3W_2}{W_1 + 0.8W_2} \right| \right)
\end{align*}

\begin{align*}
(5) \quad \text{Suppose the minimum Risk portfolio has a weight } \hat{w} = (W_1, W_2)'
\end{align*}

Since $V_0 = W_1 + 0.8W_2 > 0$, without loss of generality, we assume $W_1 + 0.8W_2 = 1$. 