[Summary]

1. The market is free of arbitrage. \( \iff \) There is a state price vector. 
\( \iff \) There exists a risk neutral probability \( Q \). 
\( \iff d < e^{rT} < u \) (in the binomial model). 
\( \cdot \quad \hat{\mathbb{Q}} = \frac{\mathbb{Q}}{q}, \quad \hat{q} \text{ discount factor} \). 
2. The arbitrage-free market is complete. \( \iff \) The risk neutral probability \( Q \) is unique.

[Exercise 1.11]

Consider

\[ S^0_0 = (1, 1)', S_a = \begin{pmatrix} 8 & 3 \\ 3 & 10 \end{pmatrix}, S^0_b = (1, 1, 1)', S_b = \begin{pmatrix} 8 & 3 & 3.001 \\ 3 & 10 & 9.999 \end{pmatrix}. \]

Is there any arbitrage opportunity for the system \( S^0_a, S_a \)? Is there any arbitrage opportunity for the system \( S^0_b, S_b \)?

\[ S_a' \hat{\mathbb{Q}} = \hat{S}_a^0 \]

\( \Rightarrow \) There is positive sol.

\( \Rightarrow \) The state price vector exists.

\( \Rightarrow \) no arbitrage opportunity.

\[ S_b' \hat{\mathbb{Q}} = \hat{S}_b^0 \]

\( \Rightarrow \hat{q} = (0.25, 0.25, -0.25)' \)

\( \Rightarrow \) The state price vector does not exist.

\( \Rightarrow \) arbitrage opportunity.
[Exercise 1.14]

Find the replicating portfolio for $F$ in Example 1.2. If $F^0 = 5$, state how you take the arbitrage opportunity.

1. $S\hat{w} = \hat{F}$

\[
\begin{pmatrix} 1.2 & 1 \end{pmatrix} \begin{pmatrix} \hat{w} \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}
\]

$\Rightarrow \hat{w} = \begin{pmatrix} -0.741 \\ 0.889 \end{pmatrix}$

$F^0 = S^0\hat{w} = 0.148$

2. "buy low, sell high"
short the option (\$5)
buy bond (\$4)

\[
\begin{aligned}
\text{no initial investment} \\
\text{buy the second asset (\$1)}
\end{aligned}
\]

At time $T$:

Scenario 1: $-8 + 4.8 + 10 = 6.8$

Scenario 2: $0 + 4.8 + 1 = 5.8$
[Exercise 1.16]

Consider the eight combinations of the following three pairs of disjoint events:

• \( \{m > n\}, \{m < n\} \).

• \( \{ \text{the market is complete}\}, \{ \text{the market is not complete}\} \).

• \( \{ \text{no arbitrage opportunity exists}\}, \{ \text{arbitrage opportunity does exist}\} \).

For each of the eight cases, give an example of the market in terms of \( S, F \), and \( S^0 \), with columns of \( S \) being linearly independent.

1. non-complete & arbitrage-free
   
   \( m = 2 \), \( n = 1 \), 
   \( S = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \), 
   \( S_0 = 5 \).

   \( S' \hat{q} = S_0 \Rightarrow 4_1 + 24_2 = 5 \Rightarrow \text{The state price vector exists \Rightarrow arbitrage-free} \).

   \( \hat{q} \) is not unique \Rightarrow non-complete.

2. non-complete & arbitrage:
   
   \( m = 3 \), \( n = 2 \), 
   \( S = \begin{pmatrix} 1.2 & 3 \\ 1.2 & 4 \end{pmatrix} \), 
   \( S_0 = \begin{pmatrix} 1 \end{pmatrix} \).

   \( S' \hat{q} = S_0 \).

   \( \Rightarrow 34_2 + 64_3 = -2 \).

   \( \Rightarrow \text{The state price vector does not exist} \).

   \( \Rightarrow \text{arbitrage}. \).
Consider a future contract with strike price $K = 105$ and maturity $T = 1$. Suppose the initial price of the underlying stock $S_0 = 100$ and the one-period risk-free rate is $r = 30\%$. The factor $u$ and $d$ are 1.2 and 0.9 respectively. If we model the dynamic by the one-period binomial tree model, then prove that arbitrage exists by

- constructing a portfolio explicitly;
- using Theorem 1.6;
- using Theorem 1.7.

1. The fair price of a futures contract is $K = S_0 e^{rT}$
   $100 \times e^{0.3} = 134.986$

   - If $K > S_0 e^{rT}$, then borrow $S_0$, buy spot and short the futures.
     At $T$, deliver the underlying, get $K$, pay back $S_0 e^{rT}$, and receive the risk-free gain of $k - S_0 e^{rT} > 0$.

   - If $K < S_0 e^{rT}$, then short the underlying, put $S_0$ into bank and long the futures.

   "buy low, sell high"

2. $S_0 = (100, 100)'$, $S = \begin{pmatrix} 134.986 & 120 \\ 134.986 & 90 \end{pmatrix}$
   $\tilde{F} = (15, -15)'$
   $\tilde{S}'\tilde{q} = \tilde{S}_0$ $\Rightarrow$ no positive sol

   $\Rightarrow$ The state price vector does not exist

   $\Rightarrow$ arbitrage opportunity.