[Definition] (Sample space)
The sample space \( \Omega \) is the collection of all possible outcomes of a particular experiment. An element in the sample space is called a sample point and is denoted by \( \omega \). A subset of the sample space is called an event.

[Definition] (\( \sigma \)-field)
A collection of events of \( \Omega \) is called a \( \sigma \)-field, denoted by \( \mathcal{F} \), if it satisfies the following three properties:

(1) \( \emptyset \in \mathcal{F} \);

(2) If \( A \in \mathcal{F} \), then \( A^c \in \mathcal{F} \);

(3) If \( A_1, A_2, \ldots \in \mathcal{F} \), then \( \bigcup_{i=1}^{\infty} A_i \in \mathcal{F} \).
[Exercise 2.9]

Explain whether the followings are $\sigma$-fields or not:

(1) Suppose $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{F} = \{\{1, 2\}, \{3, 4, 5\}, \{6\}\}$.

(2) Suppose $\Omega$ and $\mathcal{F}$ are defined as in part (1). Let $\mathcal{F}^* = \{A : A$ is the union of some subsets in $\mathcal{F}\}$. Is $\mathcal{F}^*$ a $\sigma$-field?
[Definition] (σ-field generated by a set)

Let $\mathcal{A}$ be a collection of subsets of $\Omega$, $\sigma(\mathcal{A})$ is the smallest $\sigma$-field containing $\mathcal{A}$.

[Exercise 2.13]

Show that the $\sigma$-field generated by a class of sets $\mathcal{A}$ can be expressed as

$$\sigma(\mathcal{A}) = \cap_{\alpha} \mathcal{F}_\alpha,$$

where $\mathcal{F}_\alpha$ are all the $\sigma$-fields (possibly uncountable) that contain $\mathcal{A}$.  

[Definition] (Borel $\sigma$-field)

Let $C$ be the collection of all finite open intervals on $\mathbb{R}$, then

$$B = B_{\mathbb{R}} = \sigma(C)$$

is called the Borel $\sigma$-field, whose element is called Borel set. Particularly, the Borel $\sigma$-field on $[a,b]$ is denoted by

$$B_{[a,b]} = \{ [a,b] \cap B : B \in B \}.$$

[Exercise 2.16]

Show that $B_{[a,b]}$ is a $\sigma$-field for any $-\infty < a < b < \infty$. 
[Exercise 2.8]

Let $-\infty < a < b < \infty$. Show that the following sets belong to the Borel $\sigma$-field:

1. Singleton: $\{x\}$, where $x \in \mathbb{R}$.

2. Half closed interval: $[a, b)$ or $(a, b]$.

3. Closed interval: $[a, b]$.

4. $(-\infty, b)$ and $(a, \infty)$. 
[Exercise 2.15]

Show that any continuous function \( f : \mathbb{R} \to \mathbb{R} \) is a Borel Measurable.