Chapter 6
Ito’s Stochastic Calculus

6.1 Introduction

When Bachelier first applied Wiener process on modeling the fluctuation of asset prices, the price of an asset at time \( t \), \( X_t \), has an infinitesimal increment \( dX_t \) proportional to the increment \( dW_t \) of the Wiener process, i.e.,

\[
dX_t = \sigma dW_t,
\]

where \( \sigma \) is a positive constant. As a result, an asset with initial price \( X(0) = x \) worths

\[
X_t = x + \sigma W_t
\]
at time \( t \). This model suffers from one serious flaw: for any \( t > 0 \) the price \( X_t \) can be negative with non-zero probability (but actual stock prices are never negative). To tackle this problem, successors assumed that the relative price \( dX_t / X_t \) of an asset is proportional to \( dW_t \), i.e.,

\[
dX_t = \sigma X_t dW_t, \quad (6.1)
\]

Although this equation looks like a differential equation, traditional methods are no longer applicable because the paths of \( W_t \) are not differentiable (Theorem 5.2). A way around the obstacle was found in the 1940s by Ito, who gave a rigorous meaning to (6.1) by writing it as

\[
X_t = x + \sigma \int_0^t X_s dW_s, \quad (6.2)
\]

where the integral with respect to \( W_t \) on the right-hand side is called the \textit{Ito stochastic integral}. In this section we discuss the definition and properties of stochastic integral, and explore the applications to pricing financial products.
6.2 Ito Stochastic Integral

6.2.1 Motivation

Motivated by (6.2), we construct the Ito stochastic integral in the form \( \int_0^t f(s) dW_s \) for some stochastic process/random function \( f(s, \omega) \), to be precise. We follow an approach similar to constructing Riemann integral, i.e., define the integral by the limit of the discretized version

\[
\sum_{i=0}^{n-1} f(s_i)(W_{t_{i+1}} - W_{t_i}), \quad (6.3)
\]

where \( s_i \in [t_i, t_{i+1}] \). The major differences between Riemann and Ito integrals are

1. Riemann integration results in a real number, but Ito integration results in a random variable (since \( W_t \) is random). Thus, while defining Riemann integral involves convergence of real numbers, defining Ito integral in (6.3) requires convergence of random variables, which is considerably more difficult.

2. If a Riemann integral exists, then \( s_i \) can be an arbitrary point in \([t_i, t_{i+1}]\) since the upper and lower Riemann sum converge. However, in Ito integral, the limit will be different depending on the choice of \( s_i \). This is due to the non-zero quadratic variation of the Brownian motion \( W_t \). See Exercise 6.1.

To avoid the ambiguity in 2), the definition of stochastic integrals will fix the choice \( s_i = t_i \) for each \( i \) in the approximating sum (6.3). The choice \( s_i = t_i \) is natural if we regard \( f(t) \) as the trading strategy and \( W_t \) as the stock price: For the \( i+1 \)-th period \([t_i, t_{i+1}]\), the trading strategy should only depend on the information up to time \( t_i \). Hence, \( f(t_i) \) units are invested and a profit of \( f(t_i)(W_{t_{i+1}} - W_{t_i}) \) is made. Therefore, \( \sum_{i=0}^{n-1} f(t_i)(W_{t_{i+1}} - W_{t_i}) \) represents the total profit.

Remark 6.1. (Previsible) To be precise, in defining stochastic integral we require the integrand \( f(t) \) to be previsible or predictable, i.e., \( f(t) \) is \( \mathcal{F}_t \) for all \( t \) where \( \mathcal{F}_{t^-} \equiv \bigcup_{s < t} \mathcal{F}_s \) and \( \mathcal{F}_t = \sigma(\{W_s, s \leq t\}) \). However, it can be shown that if \( f(t) \) is continuous and adapted to \( \mathcal{F}_t \), then \( f(t) \) is automatically previsible. Since we mainly deal with continuous integrand, we do not distinguish between previsible and adapted process.

6.2.2 Ito Integral

Ito integral is a random variable since \( W_t \) and the integrand \( f(t) \) are random. To ensure regularity of the Ito integral (such as the existence of the first and the second moments), we restrict \( f(t) \) to the following class of stochastic processes.

Definition 6.1. (\( \mathcal{M}^2_T \) and \( \mathcal{M}^2 \) Stochastic Processes) Denote \( \mathcal{M}^2_T \) to be the class of stochastic processes \( f(t), t \geq 0 \), such that
6.2 Ito Stochastic Integral

\[ \mathbb{E} \left( \int_0^T |f(t)|^2 \, dt \right) < \infty. \]

Let \( \mathcal{M}^2 \) be the class of stochastic processes \( f(t) \) such that \( f(t) \in \mathcal{M}^2_T \) for any \( T > 0 \). Recall that a random variable \( X \) is in \( L^2 \), or \( X \in L^2 \), if \( \mathbb{E}|X|^2 < \infty \). Both \( \mathcal{M}^2 \) and \( L^2 \) are related to the existence of second moment, but \( \mathcal{M}^2 \) is for a stochastic process and \( L^2 \) is for a random variable.

Since the Ito integral is a random variable and the integrand is a random function (stochastic process), we need to define a measure of length and a mode of convergence in defining the integral by a limit of a discretization in (6.3).

**Definition 6.2. (\( L^2 \) and \( \mathcal{M}^2 \) Norms)** For a random variable \( X \) and a stochastic process \( f \equiv f(t) \), the \( L^2 \) and \( \mathcal{M}^2_T \) norm are given respectively by

\[
\|X\|_{L^2} = \sqrt{\mathbb{E}(X^2)} \quad \text{and} \quad \|f\|_{\mathcal{M}^2_T} = \sqrt{\mathbb{E} \left( \int_0^T |f(t)|^2 \, dt \right)}. \tag{6.4}
\]

**Definition 6.3. (\( L^2 \), \( \mathcal{M}^2 \) and \( \mathcal{M}^2 \) Convergences)** A sequence of random variables \( \{X_n\} \) converges in \( L^2 \) to \( X \) if

\[
\|X_n - X\|_{L^2} = \sqrt{\mathbb{E}(|X_n - X|^2)} \to 0.
\]

as \( n \to \infty \). A sequence of random functions / stochastic processes \( \{f_n(t)\} \) converges in \( \mathcal{M}^2_T \) to \( f \) if for any \( T \),

\[
\|f_n - f\|_{\mathcal{M}^2_T} = \sqrt{\mathbb{E} \left( \int_0^T |f_n(t) - f(t)|^2 \, dt \right)} \to 0.
\]

Similarly, \( \{f_n(t)\} \) converges in \( \mathcal{M}^2 \) to \( f \) if \( \{f_n(t)\} \) converges to \( f \) in \( \mathcal{M}^2_T \) for all \( T \).

Using Definitions 6.1 and 6.3, we define the Ito Integral on the class of stochastic process \( \mathcal{M}^2 \) as follows:

**Definition 6.4. (Ito Integral)** For any \( T > 0 \) and any stochastic process \( f \in \mathcal{M}^2 \), the stochastic integral of \( f \) on \([0, T]\) is defined by

\[
I_T(f) = \lim_{n \to \infty} \sum_{j=0}^{n-1} f(t_j) (W(t_{j+1}) - W(t_j)) \, ,
\]

where \((0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T)\) is any partition of \([0, T]\) with \( \max_j |t_j - t_{j-1}| \to 0 \) as \( n \to \infty \). We write \( I_T(f) \) as

\[
I_T(f) = \int_0^T f(t) \, dW_t, \quad \text{or} \quad \int_0^T f \, dW.
\]
The following theorem justifies the above definition of Ito integral.

**Theorem 6.1. (Existence and Uniqueness of Ito Integral)** Suppose that a function $f \in \mathcal{M}^2$ satisfies the following assumptions: For all $t \geq 0$,

A1. $f(t)$ is almost surely continuous, i.e., $\mathbb{P}(\lim_{\epsilon \to 0} |f(t+\epsilon) - f(t)| = 0) = 1$,

A2. $f(t)$ is adapted to the filtration $\{\mathcal{F}_t\}$, where $\mathcal{F}_t = \sigma(\{W_s, s \leq t\})$.

Then, for any $T > 0$, the Ito integral

$$I_T(f) = \int_0^T f(t) \, dW_t$$

exists and is unique almost everywhere.

**Proof.** The proof proceeds in three steps:

1) Construct a sequence of adapted stochastic processes $f_n$ such that $\|f - f_n\|_{M^2} \to 0$.

2) Show that $\|I_T(f_n) - I_T(f)\|_{L^2} \to 0$.

3) Show the a.s. uniqueness of the limit $I_T(f)$.

To begin,

1) First we find a sequence of $\mathcal{M}^2$ functions $f_1, f_2, \ldots$ such that $\|f - f_n\|_{M^2} = E(\int_0^T |f_n(t) - f(t)|^2 \, dt) \to 0$. Define

$$f_n(t) = \begin{cases} n \int_{k-1/n}^{k/n} f(s) \, ds & \text{if } t \in [k/n, (k+1)/n) \text{ for } k = 1, 2, \ldots, \lfloor Tn \rfloor - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

Then by construction $f_n(t)$ is adapted to $\mathcal{F}_t$ and is a step function for each $\omega$. (Function of this kind is called random step function. Since it is random, to be precise, it should be denoted by $f_n(t) = f_n(t, \omega)$). Note that

$$\int_{k/n}^{(k+1)/n} |f_n(t)|^2 \, dt = n \left| \int_{k-1/n}^{k/n} f(t) \, dt \right|^2 \leq \int_{k-1/n}^{k/n} |f(t)|^2 \, dt, \text{a.s.} \tag{6.6}$$

by Cauchy Schwartz (CS) inequality. The above inequality holds almost surely as CS is applied for each $\omega$ in $f(t, \omega)$.

Next we show $\|f - f_n\|_{M^2} = E(\int_0^T |f_n(t) - f(t)|^2 \, dt) \to 0$. By the a.s. continuity of $f(t)$, it can be shown that

$$\lim_{n \to \infty} \int_0^T |f_n(t) - f(t)|^2 \, dt = 0 \text{ a.s.} \tag{6.7}$$

Let $Y_n = \int_0^T |f_n(t) - f(t)|^2 \, dt$. Note that (6.7) means that $\lim_{n \to \infty} Y_n = 0$. Note also that
\[ Y_n \leq 2 \int_0^T (|f(t)|^2 + |f_n(t)|^2) \, dt \leq 4 \int_0^T |f(t)|^2 \, dt \triangleq \bar{Y}, \]

where the second inequality follows from summing over all \( k \) in (6.6). Since \( Y_n \overset{a.s.}{\to} 0 \), \( |Y_n| \leq \bar{Y} \) a.s. and \( E\bar{Y} < \infty \) by the definition of \( \mathcal{M}_T^2 \), we can apply Dominant Convergence Theorem (DCT) to obtain \( E(Y_n) \to 0 \), i.e.,

\[ \|f - f_n\|_{\mathcal{M}_T^2} = E\left( \int_0^T |f_n(t) - f(t)|^2 \, dt \right) \to 0. \quad (6.8) \]

2) To show the existence of Ito Integral, we need to show that \( I_T(f_n) = \int_0^T f_n(t) \, dW_t \) converges to an element in \( \mathcal{L}^2 \). Since \( f_n(t) \) is a step function taking constant values in each interval \( \left[ \frac{k}{n}, \frac{k+1}{n} \right) \), we can write

\[ I_T(f_n) = \sum_{k \geq 1} f_n \left( \frac{k}{n} \right) \left( W_{\frac{k+1}{n}} - W_{\frac{k}{n}} \right). \]

Note that

\[
\|I_T(f_n)\|^2_{\mathcal{L}^2} = E \left( \sum_{k \geq 1} f_n \left( \frac{k}{n} \right) \left( W_{\frac{k+1}{n}} - W_{\frac{k}{n}} \right) \right)^2
\]

\[
= E \left( \sum_{k \geq 1} \sum_{j \geq 1} f_n \left( \frac{k}{n} \right) f_n \left( \frac{j}{n} \right) \left( W_{\frac{k+1}{n}} - W_{\frac{k}{n}} \right) \left( W_{\frac{j+1}{n}} - W_{\frac{j}{n}} \right) \right)
\]

\[
= \sum_{k \geq 1} E \left( f_n \left( \frac{k}{n} \right)^2 \right) E \left( W_{\frac{k+1}{n}} - W_{\frac{k}{n}} \right)^2 \quad \text{(Independent increment and } f_n \left( \frac{k}{n} \right) \in \mathcal{F}_n \text{)}
\]

\[
= \sum_{k \geq 1} E \left( f_n \left( \frac{k}{n} \right)^2 \right) \frac{1}{n} \quad \text{(Stationary increment and } E(W_i^2) = t) \]

\[
= E \int_0^T f_n(t)^2 \, dt \quad \text{(Riemann integral)}
\]

\[
= \|f_n\|^2_{\mathcal{M}_T^2}. \quad (6.9)
\]

Similarly, it can be shown that

\[
\|I_T(f_n) - I_T(f_m)\|^2_{\mathcal{L}^2} = \|f_n - f_m\|^2_{\mathcal{M}_T^2}. \quad (6.10)
\]

From (6.8), for any \( \varepsilon > 0 \), there is an \( N \) such that \( \|f - f_n\|_{\mathcal{M}_T^2} < \varepsilon \) for all \( n > N \).

Thus for \( n, m > N \),
\[ \| I_T(f_n) - I_T(f_m) \|_{\mathcal{H}^2}^2 = \| f_n - f_m \|_{\mathcal{H}^2}^2 \]

\[ \leq \| f_n - f \|_{\mathcal{H}^2}^2 + \| f_m - f \|_{\mathcal{H}^2}^2 \quad \text{(Triangular Inequality)} \]

\[ < \varepsilon + \varepsilon = 2 \varepsilon. \]

The sequence \( \{ I_T(f_n) \}_{n \geq 1} \) with the property \( \| I_T(f_n) - I_T(f_m) \|_{\mathcal{H}^2} \to 0 \) for \( n, m \to \infty \) is called a (Cauchy sequence). It is well known in mathematical analysis that any Cauchy sequence in \( \mathcal{H}^2 \) has a limit. Call the limit \( I_T(f) \).

3) Finally, we show the uniqueness of the limit. Suppose that there are two sequences of stochastic processes \( \{ f_i^{(1)} \}_{i \geq 1} \) and \( \{ f_i^{(2)} \}_{i \geq 1} \) satisfying \( E(\int_0^T |f(t) - f_n|^2 dt) \xrightarrow{n \to \infty} 0 \) for \( j = 1, 2 \). We need to show that \( I_T(f_n^{(j)}) \), \( j = 1, 2 \), converge to the same limit. Introduce a new sequence

\[ \{ g_i \}_{i \geq 1} = \{ f_1^{(1)}, f_1^{(2)}, f_2^{(1)}, f_2^{(2)}, f_3^{(1)}, \ldots \}. \]

By construction, \( \{ g_i \}_{i \geq 0} \) satisfies \( E(\int_0^T |f(t) - g_n|^2 dt) \xrightarrow{n \to \infty} 0 \). Therefore, the argument in 2) shows that \( I_T(g_n) \) converges to some limit. Note that if a sequence converges, then every subsequence converges to the same limit (Exercise 6.2). Thus \( I_T(f_n^{(1)}) \) and \( I_T(f_n^{(2)}) \) do converge to the same limit, completing the proof of uniqueness.

Note that the above uniqueness is proved in \( \mathcal{L}^2 \) sense, i.e., \( \| I_T(f^{(1)}) - I_T(f^{(2)}) \|_{\mathcal{L}^2}^2 = 0 \), since

\[ \| I_T(f^{(1)}) - I_T(f^{(2)}) \|_{\mathcal{H}^2}^2 \leq \| I_T(f^{(1)}) - I_T(g_n) \|_{\mathcal{H}^2}^2 + \| I_T(g_n) - I_T(f^{(2)}) \|_{\mathcal{H}^2}^2 \]

for all \( n \) and the quantity on the right is arbitrarily small. However, \( \| I_T(f^{(1)}) - I_T(f^{(2)}) \|_{\mathcal{L}^2}^2 = 0 \) implies that \( I_T(f^{(1)}) = I_T(f^{(2)}) \) almost surely (See Exercise 6.13). Thus the definition of Ito’s integral is unique a.s..

\[ \square \]

**Example 6.1.** To show the existence of \( \int_0^T W_t \, dW_t \), from Theorem 6.1 we need to show that the Wiener process \( W_t \) belongs to \( \mathcal{M}^2 \). Since for all \( T \)

\[ E(\int_0^T |W_t|^2 dt) = \int_0^T E(|W_t|^2) dt = \int_0^T t \, dt < \infty. \]

Thus \( W_t \) belongs to \( \mathcal{M}^2 \). Also, we have seen from Chapter 5 that \( W_t \) satisfies Assumptions A1 and A2 of Theorem 6.1. Hence the existence of the Ito integral \( \int_0^T W_t \, dW_t \) is justified.

**Example 6.2.** We derive the formula \( \int_0^T W_t \, dW_t = \frac{1}{2} W_T^2 - \frac{1}{2} T \) directly from definition, i.e., by approximating the integrand by random step functions. Fix \( T > 0 \) and \( t_i^n = \frac{iT}{n} \), set
6.3 Properties of the Stochastic Integral

\[ f_n(t) = \sum_{i=0}^{n-1} W^n_i 1_{[t^n_i, t^n_{i+1})}(t). \]

Then the sequence \( f_1, f_2, \cdots \in \mathcal{H}_T^2 \) approximates \( f \), since

\[
\mathbb{E} \left( \int_0^\infty |f(t) - f_n(t)|^2 \, dt \right) = \sum_{i=0}^{n-1} \int_{t^n_i}^{t^n_{i+1}} \mathbb{E} \left( |W_t - W^n_t|^2 \right) \, dt \\
= \sum_{i=0}^{n-1} \int_{t^n_i}^{t^n_{i+1}} (t - t^n_i) \, dt \\
= \frac{1}{2} \sum_{i=0}^{n-1} (t^n_{i+1} - t^n_i)^2 \\
= \frac{1}{2} T^2 \frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty.
\]

From Theorem 6.1, \( \lim_{n \to \infty} I_T(f_n) \) exists in \( \mathcal{L}^2 \) sense. To find an explicit formula of the limit, note from the equation \( a(b - a) = \frac{1}{2} (b^2 - a^2) - \frac{1}{2} (b - a)^2 \) that

\[
I_T(f_n) = \sum_{i=0}^{n-1} W^n_i \left( W^{n}_{t^n_{i+1}} - W^n_i \right) \\
= \frac{1}{2} \left[ \sum_{i=0}^{n-1} \left( W^n_{t^n_{i+1}} - W^n_i \right)^2 - \sum_{i=0}^{n-1} (W^n_{t^n_{i+1}} - W^n_i)^2 \right] \\
= \frac{1}{2} W_T^2 - \frac{1}{2} \sum_{i=0}^{n-1} (W^n_{t^n_{i+1}} - W^n_i)^2 \to \frac{1}{2} W_T^2 - \frac{1}{2} T
\]

in \( \mathcal{L}^2 \) as \( n \to \infty \). The last convergence follows from the fact that the quadratic variation of Wiener process is \( T \). Therefore, we conclude that

\[
\int_0^T W_t \, dW_t = I_T(f) = \frac{1}{2} W_T^2 - \frac{1}{2} T.
\]

\( \Box \)

6.3 Properties of the Stochastic Integral

The basic properties of the Ito integral are summarized in the following theorem:

**Theorem 6.2.** The following properties hold for any \( f, g \in \mathcal{H}_T^2 \), any \( \alpha, \beta \in \mathbb{R} \) and any \( 0 \leq s < T \):

a) **Linearity:**
\[ \int_0^T (\alpha f(t) + \beta g(t)) \, dW(t) = \alpha \int_0^T f(t) \, dW(t) + \beta \int_0^T g(t) \, dW(t); \quad (6.11) \]

b) **Isometry:**

\[ \mathbb{E}\left( \left| \int_0^T f(t) \, dW(t) \right|^2 \right) = \mathbb{E}\left( \int_0^T |f(t)|^2 \, dt \right); \quad (6.12) \]

c) **Martingale Property:**

\[ \mathbb{E}\left( \left| \int_0^T f(t) \, dW(t) \right| \mathcal{F}_t \right) = \int_0^T f(t) \, dW(t). \]

In particular, \( \mathbb{E}\left( \int_0^T f(t) \, dW(t) \right) = 0. \)

**Proof.**

a) If \( f \) and \( g \) belong to \( \mathcal{M}^2 \), then from the proof of Theorem 6.1, they can be approximated by some sequences \( f_1, f_2, \ldots \) and \( g_1, g_2, \ldots \). From Definition 6.4, it is clear that, for each \( n \),

\[ I(\alpha f_n + \beta g_n) = \sum_{j=0}^{n-1} \left( \alpha f(t^*_j) + \beta g(t^*_j) \right) \left( W_{t^*_{j+1}} - W_{t^*_{j}} \right) = \alpha I(f_n) + \beta I(g_n), \]

where \( \{t^*_j\}_{j=0,1,\ldots,n} \) is the “common” grid where both \( f_n \) and \( g_n \) are constant in each interval \((t^*_{j-1}, t^*_j)\). Taking limit on both sides of this equality as \( n \to \infty \), we obtain

\[ I(\alpha f + \beta g) = \alpha I(f) + \beta I(g), \]

which is (6.11).

b) The word **isometry** means an equality under different metrics. Note that (6.12) can be written as \( \|I_T(f)\|_{L^2} = \|f\|_{\mathcal{M}^2} \). Hence, the Ito integral connects the two metrics \( L^2 \) and \( \mathcal{M}^2 \). The proof of (6.12) follows by constructing the \( f_n \) in (6.5) such that \( \|I_T(f_n) - I_T(f)\|_{L^2} \to 0 \), \( \|f_n - f\|_{\mathcal{M}^2} \to 0 \). Combining with (6.9) in the proof of Theorem 6.1, the result follows.

c) Following the proof of Theorem 6.1, we can find a random step function \( f_n \) approximating \( f \) such that \( \|I_T(f_n) - I_T(f)\|_{L^2} \to 0 \). W.L.O.G., we can assume \( s = t_k \) for some \( k \). (adding an extra point still gives a step function). Thus
6.4 Ito’s Lemma

In this section we prove the Ito’s Lemma which is the foundation of mathematical finance and stochastic calculus.

6.4.1 The case $F(t, W_t)$

**Theorem 6.3. (Ito’s Lemma)** Suppose that $F(t, x)$ is a real valued function with continuous partial derivatives $F_t(t, x)$, $F_x(t, x)$ and $F_{xx}(t, x)$ for all $t \geq 0$ and $x \in \mathbb{R}$. Assume also that the process $F_x(t, W_t)$ belongs to $\mathcal{M}^2$. Then $F(t, W_t)$ satisfies

$$F(T, W_T) - F(0, W_0) = \int_0^T \left[ F_t(t, W_t) + \frac{1}{2} F_{xx}(t, W_t) \right] dt + \int_0^T F_x(t, W_t) dW_t, \quad \text{a.s.} \quad (6.15)$$
In differential notation, (6.15) can be written as

\[ dF(t, W_t) = \left[ F_t(t, W_t) + \frac{1}{2} F_{xx}(t, W_t) \right] dt + F_x(t, W_t) dW_t. \]  \hspace{1cm} (6.16)

**Remark 6.4**

*a) Compare (6.16) with the usual chain rule

\[ dF(t, x_t) = F_t(t, x_t) dt + F_x(t, x_t) dx_t \]

for a differentiable function \( x_t \). The additional term \( \frac{1}{2} F_{xx}(t, W_t) \) in (6.16) is called the Ito correction.

*b) Equation (6.16) is often written in the abbreviated form

\[ dF = \left( F_t + \frac{1}{2} F_{xx} \right) dt + F_x dW_t. \]  \hspace{1cm} (6.17)

Strictly speaking, the differential notation (6.16) does not make sense due to the non-differentiability of Brownian motion (\( dW_t \) is undefined).

**Proof.** We first prove the case where \( F, F_t, F_{xx} \) are all bounded by some \( C > 0 \). Consider a partition \( 0 = t_0^n < t_1^n < \cdots < t_n^n = T \) of \([0, T]\), where \( t_i^n = \frac{iT}{n} \). Denote \( W_{t_i^n} \) by \( W_i^n \); the increments \( W_{i+1}^n - W_i^n \) by \( \Delta_i^n W_i \); and \( t_i^n - t_{i-1}^n \) by \( \Delta_i^n t_i \). Using Taylor’s expansion, there is a point \( \tilde{W}_i^n \) in each interval \([W_i^n, W_{i+1}^n]\) and a point \( \tilde{t}_i^n \) in each interval \([t_{i-1}^n, t_i^n]\) such that

\[ F(T, W_T) - F(0, W_0) = \sum_{i=0}^{n-1} \left[ F(t_{i+1}^n, W_{i+1}^n) - F(t_i^n, W_i^n) \right] \]

\[ = \sum_{i=0}^{n-1} \left[ F(t_{i+1}^n, W_{i+1}^n) - F(t_i^n, W_i^n) \right] + \sum_{i=0}^{n-1} \left[ F(t_i^n, W_i^n) - F(t_i^n, \tilde{W}_i^n) \right] \]

\[ = \sum_{i=0}^{n-1} F_t(t_i^n, W_i^n) \Delta_i^n t_i + \sum_{i=0}^{n-1} F_x(t_i^n, W_i^n) \Delta_i^n W_i + \frac{1}{2} \sum_{i=0}^{n-1} F_{xx}(t_i^n, W_i^n) (\Delta_i^n W_i)^2 \]

\[ + \frac{1}{2} \sum_{i=0}^{n-1} F_{xx}(t_i^n, \tilde{W}_i^n) (\Delta_i^n W_i)^2 \]

\[ = A_{1,n} + A_{2,n} + A_{3,n} + A_{4,n} + A_{5,n}, \]  \hspace{1cm} (6.18)

say. Note that as \( F, F_t, F_{xx} \) are continuous and bounded functions, we have
Now we deal separately with each sum in (6.18):

i) From the continuity of $F_t$ and $F_{xx}$ in (6.19) and (6.20), we have the convergence of the Riemann integral

$$\lim_{n \to \infty} A_{i,n} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} F_{i} \left( \frac{t_{i}}{n}, W_{i+1}^{n} \right) \Delta^n t = \int_0^T F_t(t, W_t) \, dt \quad \text{a.s.},$$

and

$$\lim_{n \to \infty} A_{3,n} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} F_{xx} \left( t_{i}, W_{i}^{n} \right) \Delta^n t = \int_0^T F_{xx}(t, W_t) \, dt \quad \text{a.s.}$$

ii) From the assumption $F_t \in \mathcal{H}^2$, we have

$$\lim_{n \to \infty} A_{3,n} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} F_{xx} \left( t_{i}, W_{i}^{n} \right) \Delta^n W = \int_0^T F_{xx}(t, W_t) \, dW_t$$

in $\mathcal{L}^2$ from Theorem 6.1.

iii) We show the $\mathcal{L}^2$ convergence $E(A_{4,n}^2) \to 0$. To be specific,

$$E(A_{4,n}^2) = E \left( \sum_{i=0}^{n-1} F_{xx} \left( t_{i}, W_{i}^{n} \right) \left( (\Delta^n W)^2 - \Delta^n t \right) \right)^2$$

$$= \sum_{i=0}^{n-1} E \left[ F_{xx} \left( t_{i}, W_{i}^{n} \right) \left( (\Delta^n W)^2 - \Delta^n t \right) \right]^2 \quad \text{(cross terms have expectation 0)}$$

$$= \sum_{i=0}^{n-1} E \left[ F_{xx} \left( t_{i}, W_{i}^{n} \right) \right]^2 E \left( (\Delta^n W)^2 - \Delta^n t \right)^2 \quad \text{(independent increment)}$$

$$\leq C^2 \sum_{i=0}^{n-1} E \left( (\Delta^n W)^2 - \Delta^n t \right)^2 \quad \text{(boundedness of } F_{xx} \text{)}$$

$$= 2C^2 \sum_{i=0}^{n-1} (\Delta^n t)^2 = 2C^2 \frac{\sum_{i=0}^{n-1} t_i^2}{n^2} = 2C^2 \frac{T^2}{n} \to 0 \quad \text{as } n \to \infty.$$
\[ |A_{5,n}| = \left| \sum_{i=0}^{n-1} \left[ F_{xx}(t^n_i, \tilde{W}^n_i) - F_{xx}(t^n_i, W^n_i) \right] (\Delta^n_i W)^2 \right| \leq \sup_{i=1,2,\ldots,n} \left| F_{xx}(t^n_i, \tilde{W}^n_i) - F_{xx}(t^n_i, W^n_i) \right| \sum_{i=0}^{n-1} (\Delta^n_i W)^2 \xrightarrow{p} 0. \]

Note that the convergence of \( A_{1,n}, i = 1, \ldots, 5 \) involves different modes: \( A_{1,n} \) and \( A_{2,n} \) converge almost surely, \( A_{3,n}, A_{4,n} \) converge in \( \mathcal{L}^2 \), and \( A_{5,n} \) converges in probability. To combine the results, note from Exercise 6.15 that convergence in \( \mathcal{L}^2 \) implies convergence in probability. Thus all \( A_{3,n}, A_{4,n} \) and \( A_{5,n} \) converge in probability. Note also from Exercise 6.16 that there is a subsequence \( \{n_k\}_{k=1,2,\ldots} \) such that \( \{A_{3,n_k}\}_{k=1,2,\ldots} \) converge a.s. Along this subsequence, we can find a further subsequence \( n_{k_j} \) such that \( A_{4,n_{k_j}} \) converges a.s., and so forth. Therefore, all \( A_{j,n}, j = 1, \ldots, 5 \) converge a.s. with respect to some subsequence \( m_1 < m_2 < \cdots \), say. Then

\[
F(T, W_T) = F(0, W_0) = \lim_{k \to \infty} \left\{ \sum_{i=0}^{m_k-1} F_i(t^{m_k}_i, W^{m_k}_i) \Delta^m t + \frac{1}{2} \sum_{i=0}^{m_k-1} F_{xx}(t^{m_k}_i, W^{m_k}_i) \Delta^m t + \sum_{i=0}^{m_k-1} F_i(t^{m_k}_i, W^{m_k}_i) \Delta^m W \right\}
\]

completing the proof. The general case where \( F_i, F_x \) and \( F_{xx} \) are not bounded is deferred to Exercise 6.8.

**Example 6.3.** For \( F(t, x) = x^2 \), we have \( F_i(t, x) = 0 \), \( F_x(t, x) = 2x \) and \( F_{xx}(t, x) = 2 \). Ito formula gives \( dW^2 = dt + 2W_t dW_t \), provided that \( 2W_t \in \mathcal{M}^2 \) (which has been verified in Example 6.1). \( \square \)

### 6.4.2 General Case

Note that Ito’s Lemma aims at providing a first order approximation for \( dF(t, W_t) \) using the terms \( dt \) and \( dW_t \). In the Taylor’s expansion of \( F(t, W_t) \) up to the second order terms, the quantities \( dt, dW_t, dtdW_t, (dt)^2, (dW_t)^2, \ldots \) are involved. Since the quadratic variation of \( W_t \) is \( [W]_t = t \), it follows that the term \( (dW_t)^2 \) contributes an additional \( dt \) term. Other terms such as \( dtdW_t, (dt)^2 \) and \( (dW_t)^3 \) are smaller than \( dt \), and thus can be omitted. It is convenient to remember the results using the following Ito multiplication table:
6.4 Ito’s Lemma

<table>
<thead>
<tr>
<th>× dt dW_t</th>
</tr>
</thead>
<tbody>
<tr>
<td>dt 0 0</td>
</tr>
<tr>
<td>dW_t 0 dt</td>
</tr>
</tbody>
</table>

Table 6.1 Ito multiplication table.

Looking closely into the proof of Theorem 6.3, it can be seen that Ito’s Lemma can be extended from $F(t,W_t)$ to $F(t,X_t)$ where $X_t$ is an arbitrary process with quadratic variation $[X]_t$, satisfying $d[X]_t = g(t)dt$ for some $g(t) \in \mathcal{M}^2$. For example, if $X_t$ is an Ito Process given by

$$dX_t = a_t dt + b_t dW_t,$$

where $a_t$ and $b_t \in \mathcal{M}^2$, then it can be checked (Exercise 6.17) that $d[X]_t = b_t^2 dt$. (Informally, we can use Table 6.4.2 to see that $d[X]_t = (dX_t)^2 = a_t dt + b_t dW_t)^2 = a_t^2(dt)^2 + 2a_t b_t(dt)(dW_t) + b_t^2(dW_t)^2 = b_t^2 dt$). The following Theorem gives the general case of Ito’s Lemma for $F(t,X_t)$, where $X_t$ is an arbitrary process instead of a Brownian motion.

**Theorem 6.5. (Ito formula, general case)** Let $X_t$ be a stochastic process with quadratic variation $[X]_t$, satisfying $d[X]_t = g(t)dt$ where $g(t) \in \mathcal{M}^2$. Suppose that $F(t,x)$, $F_t(t,x)$, $F_x(t,x)$ and $F_{xx}(t,x)$ are continuous for all $t \geq 0$ and $x \in \mathbb{R}$. Also assume the process $g,F_x(t,X_t) \in \mathcal{M}^2$. Then $F(t,X_t)$ can be expressed as

$$dF(t,X_t) = F_t(t,X_t)dt + F_x(t,X_t)dX_t + \frac{1}{2}F_{xx}(t,X_t)d[X]_t.$$  

(6.23)

**Example 6.4.** If $X_t$ is an Ito process satisfying (6.22), then (6.23) reduces to (writing $F(t,X_t)$ as $F$)

$$dF = F_t dt + F_x dX_t + \frac{1}{2} F_{xx} d[X]_t,$$

$$= F_t dt + F_x (a_t dt + b_t dW_t) + \frac{1}{2} F_{xx} b_t^2 dt$$

$$= \left( F_t + F_x a_t + \frac{1}{2} F_{xx} b_t^2 \right) dt + F_x b_t dW_t.$$  

We end this section with the analog of the product rule $df(x)g(x) = f(x)dg(x) + g(x)df(x)$ in stochastic calculs. Again, there is an additional term involving $dt$ contributed by the quadratic variation process.

**Corollary 6.1. (Product Rule)** If $X_t$ and $Y_t$ are processes satisfying $dX_t = a_x(t) dt + b_x(t)dW_t$ and $dY_t = a_y(t) dt + b_y(t)dW_t$, then

$$dX_t Y_t = X_t dY_t + Y_t dX_t + b_x(t)b_y(t) dt.$$
Proof. Intuitively, for a small $\Delta t$,

$$X_{t+\Delta t}Y_{t+\Delta t} - X_tY_t = X_t(Y_{t+\Delta t} - Y_t) + Y_t(X_{t+\Delta t} - X_t) + (Y_{t+\Delta t} - Y_t)(X_{t+\Delta t} - X_t)$$

As $\Delta \to 0$, by definition, $X_{t+\Delta t}Y_{t+\Delta t} - X_tY_t \to dX_tY_t$, $X_{t+\Delta t} - X_t \to dX_t$ and $Y_{t+\Delta t} - Y_t \to dY_t$. Thus Corollary 6.1 follows by noting from Table 6.4.2 that

$$(Y_{t+\Delta t} - Y_t)(X_{t+\Delta t} - X_t) \to (a_t(t)dt + b_t(t)dW_t)(a_t(t)dt + b_t(t)dW_t) = b_t(t)b_t(t)dt.$$ 

A formal proof can be obtained by similar argument as in Theorem 6.1. $\square$

### 6.5 Stochastic Differential Equations

In this section, we consider stochastic differential equation of the form

$$dX_t = f(X_t) dt + g(X_t) dW_t,$$

with initial condition $X_0 = x_0$. The goal is to obtain an explicit formula for $X_t$.

Example 6.5. Consider the initial value problem

$$\begin{cases} dX_t = -\alpha X_t dt + \sigma dW_t, \\ X_0 = x_0 \in \mathbb{R}, \end{cases}$$

Let $F(t,x) = e^{\alpha t}x$, and thus $F(0,X_0) = x_0$. By Ito’s Lemma, we have

$$dF(t,X_t) = [\alpha e^{\alpha t} X_t - \alpha e^{\alpha t} X_t] dt + \sigma e^{\alpha t} dW_t$$

$$= \sigma e^{\alpha t} dW_t.$$ 

It is easily seen that $e^{\alpha t} \in \mathcal{M}^2$. It follows that $F(t,X_t) = x_0 + \sigma \int_0^t e^{\alpha s} dW_s$, i.e.,

$$X_t = e^{-\alpha t} F(t,X_t) = e^{-\alpha t} x_0 + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dW_s.$$ 

$\square$

Example 6.6. Suppose $X_t = x_0 e^{at + bW_t}$ for $t \geq 0$. By Ito’s lemma,

$$dX_t = d \left( x_0 e^{at + bW_t} \right)$$

$$= \left( ax_0 e^{at + bW_t} + \frac{b^2}{2} x_0 e^{at + bW_t} \right) dt + bx_0 e^{at + bW_t} dW_t$$

$$= \left( a + \frac{b^2}{2} \right) X_t dt + b X_t dW_t.$$
The problem of checking that \( bX_t \in \mathcal{M}^2 \) for all \( T > 0 \) is left to the reader in Exercise 6.18. This implies the solution of the initial value problem

\[
\begin{align*}
\left\{ \begin{array}{l}
    dX_t = \left( a + \frac{b^2}{2} \right) X_t \, dt + bX_t \, dW_t, \\
    X_0 = x_0
\end{array} \right.
\end{align*}
\]

is \( X_t = x_0 e^{at + bW_t} \).

From the above examples, to solve a SDE, we need to guess a solution and use Ito’s Lemma to verify that the solution satisfies the SDE. The following table summarizes some examples and solutions of SDE. In the table, \( c \) stands for a constant and \( \sinh(x) = (e^x - e^{-x})/2 \).

<table>
<thead>
<tr>
<th>Name</th>
<th>SDE</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ornstein-Uhlenbeck(OU) process</td>
<td>( dX_t = -\alpha X_t , dt + \sigma dW_t )</td>
<td>( ce^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} , dW_s )</td>
</tr>
<tr>
<td>Mean reverting OU</td>
<td>( dX_t = (m - \alpha X_t) , dt + \sigma dW_t )</td>
<td>( m/\alpha - (c - m)e^{-\alpha t} + \sigma \int_0^t e^{\alpha (s-1)} , dW_s )</td>
</tr>
<tr>
<td>Geometric Brownian motion</td>
<td>( dX_t = aX_t , dt + bX_t , dW_t )</td>
<td>( ce^{(a-b^2/2)t + bW_t} )</td>
</tr>
<tr>
<td>Brownian bridge</td>
<td>( dX_t = \left( 1 + X_t^2 \right)^{1/2} , dt + \left( 1 + X_t^2 \right)^{1/2} , dW_t )</td>
<td>( (c + W_t) / (1 + t) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( c^{-W_t} )</td>
</tr>
<tr>
<td></td>
<td>( dX_t = bW_t - \frac{1}{2} X_t , dt ) + \sqrt{1 - X_t^2} , dW_t )</td>
<td>( \sinh(c + W_t) )</td>
</tr>
<tr>
<td></td>
<td>( dX_t = -\frac{1}{2} X_t , dt + \frac{1}{1+t} X_t , dW_t )</td>
<td>( \left( c + W_t \right) / (1 + t) )</td>
</tr>
<tr>
<td></td>
<td>( dX_t = rd , dt + \alpha X_t , dW_t )</td>
<td>( ce^{r W_t - \frac{1}{2} \alpha^2 t} + \sqrt{1 - e^{-\alpha^2 (W_t - W_s)}} \alpha^2 (1-t) )</td>
</tr>
</tbody>
</table>

### 6.6 Three Technical Results:

In this section, we discuss three technical results that constitute the proof of the **Fundamental Theorem of Asset Pricing**:

- Levy’s Characteristic of Brownian Motion,
- Girsanov Theorem,
- Brownian Martingale Representation Theorem.

As the proofs do not help the understanding of the Fundamental Theorem of Asset Pricing, one may skip the proofs in the first reading.

#### 6.6.1 Levy’s Characteristic of Brownian Motion

Levy’s characterization allows one to verify whether a process is a Brownian motion by investigating its quadratic variation.
Theorem 6.6. (Levy’s Characteristic of Brownian Motion) Let $W_t$ be a stochastic process with natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Then, $W_t$ is a Brownian motion if and only if the following conditions hold:

1) $W_0 = 0$ a.s.;
2) $W_t$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$.
3) The quadratic variation $[W]_t = t$ for all $t \geq 0$.

Proof. Since $W_0 = 0$ a.s., it suffices to check the stationary and independent increment properties of Brownian motions. For a constant $\theta > 0$, consider the stochastic process

$$M_t = e^{\theta W_t - \frac{\theta^2}{2} t}, \quad (6.24)$$

Applying Ito’s Lemma on $M_t$, together with the assumption $[W]_t = t$, we have

$$dM_t = -\frac{\theta^2}{2} M_t dt + \theta M_t dW_t + \frac{\theta^2}{2} M_t dt = \theta M_t dW_t,$$

or $M_t = \int_0^t \theta M_s dW_s$. Since $W_t$ is a martingale, the proof of Theorem 6.2(c) implies that $M_t$ is a martingale, i.e., for $s \leq t$,

$$\mathbb{E}(M_t | \mathcal{F}_s) = M_s. \quad (6.25)$$

Substituting (6.24) to (6.25), we have

$$\mathbb{E}(e^{\theta(W_t - W_s)} | \mathcal{F}_s) = e^{\frac{1}{2} (t-s) \theta^2}. \quad (6.26)$$

Since the RHS is the m.g.f. of the normal distribution $N(0, t-s)$, we have $W_t - W_s \sim N(0, t-s)$, proving the stationary increment property.

To show independent increment, note that for $0 = t_0 < t_1 < \cdots < t_n = T$, we have from (6.26) and repeated applications of the Tower property that, for constants $\theta_i$,

$$\mathbb{E} \left( e^{\sum_{i=1}^n \theta_i (W_{t_i} - W_{t_{i-1}})} \right) = \mathbb{E} \left[ \mathbb{E} \left( e^{\sum_{i=1}^n \theta_i (W_{t_i} - W_{t_{i-1}})} | \mathcal{F}_{t_{i-1}} \right) \right]$$

$$= \mathbb{E} \mathbb{E} \left( e^{\sum_{i=1}^{n-1} \theta_i (W_{t_i} - W_{t_{i-1}})} \mathbb{E} \left( e^{\theta_n (W_{t_n} - W_{t_{n-1}})} | \mathcal{F}_{t_{n-1}} \right) \right)$$

$$= \mathbb{E} \left( e^{\theta_n (W_{t_n} - W_{t_{n-1}})} \right) \mathbb{E} \left( e^{\sum_{i=1}^{n-1} \theta_i (W_{t_i} - W_{t_{i-1}})} \right), \quad \text{(independent increment)}$$

$$= \prod_{i=1}^n \mathbb{E} \left( e^{\theta_i (W_{t_i} - W_{t_{i-1}})} \right). \quad \text{(Repeating the same argument)}$$

Since the joint m.g.f. can be factorized into product of m.g.f.s, the sequence $\{W_{t_j} - W_{t_{j-1}}\}_{j=1,\ldots,n}$ are independent random variables. Thus, $W_t$ satisfies the three conditions of Brownian motions in Definition 5.2.
6.6 Three Technical Results:

Example 6.7. Recall that we have verified in Example 5.1 that $\tilde{W}_t = c^{-1/2}W_{ct}$ is a Brownian motion. Alternatively, using Theorem 6.6, the verification can be achieved by 1) showing the martingale property and 2) showing that $|\tilde{W}| = t$. First, the martingale property is inherited from the stationary and independent increment properties of Brownian motion $W_t$. Next, the quadratic variation can be computed using the same argument in Theorem 5.14: Set $Q_n = c^{-1}\sum_{i=1}^{n}(W_{ct_i} - W_{ct_{i-1}})^2$ for partitions $\Pi_n = \{t_i\}_{i=1,\ldots,n}$ of $[0,t]$ with mesh $\delta(\Pi_n) \to 0$. Then show that $E(Q_n) \to t$ and $\text{Var}(Q_n) \to 0$.

6.6.2 Girsanov Theorem

Recall that in the discrete time world, the physical probability measure is not useful for pricing. Instead, we use the asset values to deduce a risk neutral probability measure for pricing. In the continuous time world, it is no longer possible to find the risk neutral probability by linear algebra. We need the Girsanov Theorem to change the physical measure to the risk neutral probability measure for pricing.

Theorem 6.7. (Girsanov’s Theorem) Suppose that $\{W_t\}_{t \geq 0}$ is a $\mathbb{P}$-Brownian motion with natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$, and $\{\theta_t\}_{t \geq 0}$ is a $\{\mathcal{F}_t\}$-adapted process with

$$E\left(e^{\frac{1}{2} \int_0^t \theta_s^2 \, ds}\right) < \infty.$$

Given a “drifted” Brownian motion

$$W_t^Q = W_t + \int_0^t \theta_s \, ds,$$

Then, there is a measure $\mathbb{Q}$ such that $W_t^Q$ is a standard Brownian motion under $\mathbb{Q}$. The measure $\mathbb{Q}$ is given by

$$\mathbb{Q}(A) = \int_A L_t(\omega) \mathbb{P}(d\omega),$$

for all $A \in \mathcal{F}_t$ and

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_t \equiv L_t = e^{-\frac{1}{2} \int_0^t \theta_s \, dW_s - \frac{1}{2} \int_0^t \theta_s^2 \, ds}$$

is the Radon-Nikodym Derivative of $\mathbb{Q}$ w.r.t. $\mathbb{P}$, with both $\mathbb{P}$ and $\mathbb{Q}$ are restricted on $(\Omega, \mathcal{F}_t)$. Particularly,

$$E_{\mathbb{P}}(f(W_t)) = E_{\mathbb{Q}} \left( f(W_t) \frac{1}{L_t} \right) = E_{\mathbb{Q}} \left( f(W_t) e^{\int_0^t \theta_s \, dW_s + \frac{1}{2} \int_0^t \theta_s^2 \, ds} \right)$$

$$= E_{\mathbb{Q}} \left( f \left( W_t^Q - \int_0^t \theta_s \, ds \right) e^{\int_0^t \theta_s \, dW_s^Q - \frac{1}{2} \int_0^t \theta_s^2 \, ds} \right).$$
Proof. (Heuristic) First we give a heuristic derivation to understand the Girsanov Theorem. Let $X$ be a r.v. on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{P}(X < x) = \Phi(x)$, i.e., $X \sim N(0, \sigma^2)$ under $\mathbb{P}$. In this case, the p.d.f. of $X$ is given by $f_\mathbb{P}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$. Obviously, the random variable $X + \mu$ follows $N(\mu, \sigma^2)$ under $\mathbb{P}$. If we want a measure $\mathbb{Q}$ such that $X + \mu \sim N(0, \sigma^2)$ under $\mathbb{Q}$, then the p.d.f. of $X$ under $\mathbb{Q}$, $f_\mathbb{Q}$, can be derived follows: From the definition of $f_\mathbb{Q}(\cdot)$, we have

$$P_\mathbb{Q}(X < z) = \int_{-\infty}^{z} f_\mathbb{Q}(x) \, dx. \quad (6.29)$$

On the other hand, as we require that $X + \mu \sim N(0, \sigma^2)$ under $\mathbb{Q}$, we have

$$P_\mathbb{Q}(X < z) = P_\mathbb{Q}(X + \mu < z + \mu) = \int_{-\infty}^{z+\mu} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}} \, dx = \int_{-\infty}^{\frac{z}{\sigma^2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+\mu)^2}{2\sigma^2}} \, dx. \quad (6.30)$$

Comparing (6.29) and (6.30), we have

$$f_\mathbb{Q}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x+\mu)^2}{2\sigma^2}} = e^{-\frac{1}{\sigma^2}(s\mu + \frac{1}{2}\mu^2)} f_\mathbb{P}(x).$$

In particular, the Radon-Nikodym Derivative is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{f_\mathbb{Q}(x)}{f_\mathbb{P}(x)} = e^{-\frac{1}{\sigma^2}(s\mu + \frac{1}{2}\mu^2)}. \quad (6.31)$$

When $\mu = \sigma^2 = t$, (6.31) is exactly (6.27) with $\theta_i \equiv 1$.

Consider the general case where $W_t$ is a Brownian motion under $\mathbb{P}$. Let $t_i = i/n$ and $\mathbb{P}^{(n)}$ be the finite dimension probability measure for $\{W_{t_i}\}_{i=1,\ldots,n}$. Suppose that we want a measure $\mathbb{Q}^{(n)}$ under which the drifted Brownian motion $W_t = W_t + \int_0^t \theta_i \, ds$ behaves the same as a standard Brownian motion at $\{t_i\}_{i=1,\ldots,n}$.

From Proposition 5.1, the joint p.d.f of the B.M. at the time points $\{t_i\}_{i=1,\ldots,n}$ is the same as the joint p.d.f. of $\{W_{t_i} - W_{t_{i-1}}\}_{i=1,\ldots,n}$, i.e.,

$$f_\mathbb{P}^{(n)}(x_1, \cdots, x_n) = \prod_{j=1}^n \phi \left( \frac{x_j - x_{j-1} - \theta_j}{\sqrt{n}} \right), \quad (6.32)$$

where $x_0 = 0$ and $\phi(u, \mu, \sigma^2)$ is the p.d.f. of a $N(\mu, \sigma^2)$ distribution. On the other hand, since

$$\overline{W}_{t_i} - \overline{W}_{t_{i-1}} = W_{t_i} - W_{t_{i-1}} + \int_{t_{i-1}}^{t_i} \theta_j \, ds \approx W_{t_i} - W_{t_{i-1}} + \frac{1}{n} \theta_{t_{i-1}},$$

the joint p.d.f. of $\{\overline{W}_{t_i} - \overline{W}_{t_{i-1}}\}_{i=1,\ldots,n}$ under $\mathbb{P}^{(n)}$ is the product of p.d.f.s of normal random variable plus a constant (similar to $X + \mu$).
Thus, from Theorem 6.2(3),

\[ \{ \tilde{W}_i - \tilde{W}_{i-1} \}_{i=1,\ldots,n} \] has a joint p.d.f. equal to the RHS of (6.32) under \( Q^{(n)} \), the previous argument (6.31) can be employed with 
\( x = W_i - W_{i-1}, \mu = \theta_{i-1} \) and \( \sigma^2 = \frac{1}{n} \). This yields the Radon-Nikodym Derivative (RND) of the \( Q^{(n)} \) w.r.t. \( P^{(n)} \),

\[
\frac{dQ^{(n)}}{dP^{(n)}} = \prod_{j=1}^n e^{\left(-(W_j - W_{j-1})\theta_{j-1} - \frac{1}{2} \theta^2_{j-1}\right)}
\]

\[
= e^{\left(-\sum_{j=1}^n (W_j - W_{j-1})\theta_{j-1} - \sum_1^n \frac{1}{2} \theta^2_{j-1}\right)}
\]

\[
\approx e^{-\int \theta_t dW_t - \frac{1}{2} \int \theta^2_t ds},
\]

which motivates the form of the RND in (6.27).

**Proof. (Formal Proof)** We use Levy’s Characteristic of B.M. Theorem 6.6 to show that \( \{ W^Q_t \} \) is a Brownian motion under \( Q \). We need to show that 1) \( \{ W^Q_t \} \) is a \( Q \)-martingale, and 2) the quadratic variation process of \( \{ W^Q_t \} \) is \( \{ W^Q_t \}_t = t \).

1) First, note that \( L_t \) is a \((P,\{F_t \}_{t\geq 0})\) martingale. To see this, let \( Z_t = -\int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta^2_s ds \), then \( dZ_t = -\theta_t dW_t - \frac{1}{2} \theta^2_t dt \) and \( L_t = e^{Z_t} \). Using Ito’s Lemma (6.23) on \( L_t \), we have

\[
dL_t = \frac{\partial L_t}{\partial t} dt + \frac{\partial L_t}{\partial Z_t} dZ_t + \frac{1}{2} \frac{\partial^2 L_t}{\partial Z^2_t} d[Z]_t
\]

\[
= 0 + L_t dZ_t + \frac{1}{2} L_t \theta^2_t dt
\]

\[
= -\theta_t L_t dW_t, \quad \text{ (since } dZ_t = -\theta_t dW_t - \frac{1}{2} \theta^2_t dt) \quad (6.33)
\]

which implies that \( L_t \) is a \( P \)-martingale by Theorem 6.2(c).

If the measure \( Q \) on \( (\Omega,F) \) is defined by \( Q(A) \equiv \int_A L_t dP \), then the martingale property implies that

\[
\mathbb{Q}(\Omega) \equiv \int_{\Omega} L_t dP = E_P(L_0) = L_0 = 1,
\]

i.e., \( \mathbb{Q} \) is a probability measure and \( \frac{d\mathbb{Q}}{dP} \bigg|_{t \geq 0} \equiv L_t \) is the RND of \( \mathbb{Q} \) w.r.t. \( P \) on \( (\Omega,F) \).

On the other hand, from \( W^Q_t = W_t + \int_0^t \theta_s ds \), we have \( dW^Q_t = dW_t + \theta_t dt \). Also, by Corollary 6.1 (product rule), we have

\[
d\left(W^Q_tL_t\right) = W^Q_t dL_t + L_t dW^Q_t + dW^Q_t dL_t
\]

\[
= W^Q_t (\theta_t L_t dW_t) + L_t (\theta_t dt + dW_t) + (\theta_t dt + dW_t) (\theta_t L_t dW_t)
\]

\[
= (L_t - \theta_t L_t W^Q_t) dW_t.
\]

Thus, from Theorem 6.2(3),

\[
W^Q_t L_t \text{ is a } P \text{ martingale.} \quad (6.34)
\]
It follows that $W^Q_t$ is a $Q$ martingale. To see this, note from (6.34) that $E_{F} \left( W^Q_t L_t \mid \mathcal{F}_s \right) = W^Q_t L_s$, which is the same as

$$
\int_S W^Q_t L_t dF = \int_S E_{Q}(W^Q_t L_t) dF = \int_S W^Q_t L_s dF,
$$

(6.35)

for all $S \in \mathcal{F}_s$ (definition of conditional expectation). Since $L_s = \frac{dQ}{dF} \big|_t$ is the R.N.D. of $Q$ w.r.t. $F$ on $(\Omega, \mathcal{F})$, Equation (6.35) can be expressed as

$$
\int_S W^Q_t dF = \int_S W^Q_s dF,
$$

for all $S \in \mathcal{F}_s \subset \mathcal{F}_t$, which means $E_Q(W^Q_t \mid \mathcal{F}_s) = W^Q_s$, i.e., $W^Q_t$ is a $Q$ martingale.

2) Since $W^Q_t$ is a drifted version of $W_t$, similar arguments as in the proof of Theorem 5.14 show that the quadratic variation of $W^Q_t$ under $Q$ is $t$. (Exercise 6.17).

**Example 6.8.** Consider a $(\mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ Brownian motion $\{W_t\}$. Recall that from Lemma 5.10, we can obtain the joint density of $M_t = \max_{t \geq 0} W_u$ and $W_t$, say $f_{M,W}(m, w)$. Suppose we want to find

$$
E_{\mathbb{P}}(S_t I_{\{\max_{t \geq 0} S_u > K\}}), \quad \text{where } S_t = e^{\alpha t + W_t}.
$$

(6.36)

The difficulty here is that the set $\{\max_{t \geq 0} S_u > K\} = \{\max_{t \geq 0} e^{\alpha t + W_u} > K\}$ is not directly related to $M_t = \max_{t \geq 0} W_u$. To tackle this, using Girsanov Theorem (6.28) with a $Q$ such that $W^Q_t = at + W_t$ is a $Q$ B.M., (6.36) reduces to

$$
E_{\mathbb{P}} \left( e^{\alpha t + W_t} I_{\{\max_{t \geq 0} e^{\alpha t + W_u} > K\}} \right) = E_Q \left( e^{W^Q_t} I_{\{\max_{t \geq 0} W^Q_u > K\}} e^{\alpha W^Q_t - \frac{1}{2} \alpha^2 t} \right).
$$

(6.37)

Now, in (6.37), only $W^Q_t$ and $\max_{t \geq 0} W^Q_u$ are involved, and $W^Q_t$ is a standard B.M. under $Q$. Thus $f_{M,W}(m, w)$ can be employed to compute the RHS of (6.37) by

$$
\int_{m > w} \int_{w > 0} \left( \max_{t \geq 0} k \right) e^{(1 + \alpha)w - \frac{1}{2} \alpha^2 t} f_{M,W}(m, w) \, dm \, dw.
$$

**Example 6.9.** (A direct proof of Proposition 5.12(by Li Yuxuan)) Note that “a standard Brownian motion $W_t$ hitting a upward sloping line $a + bt$” is equivalent to “a downward drifted Brownian motion $W_t - bt$ hitting level $a$”. Therefore, to find the stopping time $T_{a,b} := \inf \{ t \geq 0 : W_t = a + bt \}$, we may use Girsanov Theorem with $W^Q_t = W_t - bt$ so that $T_{a,b}$ becomes $\inf \{ t \geq 0 : W^Q_t = a \}$. Then, the result in Lemma 5.13 about stopping time of hitting a horizontal level, i.e., $E \exp \{ -\theta T_a \} = \exp \left\{ -a\sqrt{2\theta} \right\}$, can be applied. Particularly,
6.6 Three Technical Results:

\[ E\left(e^{-\theta T_{a,b}}\right) = E\left(e^{-\theta \inf\{t \geq 0 : W_t^Q = a+bt\}}\right) \]

\[ = E_Q \left(e^{-\theta \inf\{t \geq 0 : W_t^Q = a\}} \cdot e^{-b(W_t^Q - W_{T_{a,b}}) - \frac{b^2 t}{2}}\right) \quad \text{(Girsanov: } W_t^Q = W_t - bt \text{ is a } \tilde{Q} \text{ B.M., } t > T_{a,b}) \]

\[ = E_Q \left(e^{-\theta \tau} \cdot e^{-b(W_t^Q - W_{\tau}) - \frac{b^2 t}{2}}\right) \quad \text{(} \tau \triangleq T_{a,b} \triangleq \inf\{t \geq 0 : W_t^Q = a\} \text{)} \]

\[ = E_{Q} \left(e^{-\theta \tau} \cdot e^{-b(W_t^Q - W_{\tau}) - ab - \frac{b^2 \tau}{2}}\right) \quad \text{(} W_{T_{a,b}}^Q = a \text{ by definition)} \]

\[ = e^{\frac{-b^2}{2} - ab} E_{Q} \left(e^{-\theta \tau} \cdot E_{Q} \left(e^{-b(W_t^Q - W_{\tau})} \left| \tau \right\rangle\right)\right) \quad \text{(Tower property and independent increment)} \]

\[ = e^{\frac{-b^2}{2} - ab} E_{Q} \left(e^{-\theta \tau} \cdot \left(e^{\frac{-b^2}{2} (\tau - \tau)}\right)\right) \quad \text{(M.G.F. of Brownian motion)} \]

\[ = e^{\tau \cdot \left(e^{\frac{-b^2}{2} \tau}\right)} \quad \text{(Lemma 5.13)} \]

\[ = e^{-ab} e^{-a \sqrt{b^2 + 2\theta}} = e^{-a(b+\sqrt{b^2 + 2\theta})}. \]

Note that this argument is only heuristic but not rigorous: in applying Girsanov theorem (6.28), the \( f(W_t) \) is \( \mathcal{F}_t \) measurable and the time \( t \) in \( L_t \) is a fixed value. However, \( T_{a,b} \) is in general not \( \mathcal{F}_t \) measurable for any fixed \( t \) (Exercise 6.4). Also, assuming “\( t > T_{a,b} \)” is not well-defined as \( T_{a,b} \) is a random variable and there may not exist a \( t \) such that \( t > T_{a,b} \) a.s..

6.6.3 Brownian Martingale Representation Theorem

Finally, the Brownian Martingale Representation Theorem shows that any martingale can be represented by a stochastic integral involving a Brownian motion.

**Theorem 6.8. (Brownian Martingale Representation Theorem)** Let \( \{ \mathcal{F}_t \}_{t \geq 0} \) denote the natural filtration of the \( \mathbb{P} \)-Brownian motion \( \{ W_t \}_{t \geq 0} \). Let \( \{ M_t \}_{t \geq 0} \) be a \( (\mathbb{P}, \{ \mathcal{F}_t \}_{t \geq 0}) \) martingale which is square-integrable, i.e., for each \( t > 0 \),

\[ E(\left| M_t \right|^2) < \infty. \]

Then, there exists a \( \{ \mathcal{F}_t \}_{t \geq 0} \) predictable process \( \{ \theta_t \}_{t \geq 0} \) such that with \( \mathbb{P} \)-probability one,

\[ M_t = M_0 + \int_0^t \theta_s dW_s. \quad (6.38) \]

**Proof.** First, we create a class of stochastic process \( \mathcal{G} \) of the form
\[ Z_t^f = e^{\int_0^t f(s) \, dW_s} - \frac{1}{2} \int_0^t f(s)^2 \, ds, \]

where \( f(s) = \sum_{i=1}^n \beta_i 1_{(t_{i-1}, t_i]}(s), \beta_i \in \mathbb{R} \) and \( \{t_i\}_{i=0,1,\ldots,n} \) is a partition of \([0, T]\). Using Ito’s formula, it can be checked that (Exercise 6.12)

\[ Z_t^f = 1 + \int_0^t f(s) Z_s^f \, dW_s. \] (6.39)

Using theory in Hilbert space, it can be shown that any square integrable random variable can be represented by a linear combination of elements in \( \mathcal{G} \) (for details, see Chapter 12 of Steele (2001)). Specifically, any martingale \( M_t \) has a representation

\[ M_t = \sum_{j=1}^{\infty} \alpha_j Z_t^{f_j}, \]

for some constant \( \alpha_j \). Hence, from (6.39) we have

\[ M_t = \sum_{j=1}^{\infty} \alpha_j + \int_0^t \sum_{j=1}^{\infty} \alpha_j f_j(s) Z_s^{f_j} \, dW_s \]

\[ = \theta_0 + \int_0^t \theta_s \, dW_s, \]

where \( \theta_0 = \sum_{j=1}^{\infty} \alpha_j \) and \( \theta_t = \sum_{j=1}^{\infty} \alpha_j f_j(s) Z_s^{f_j} \). By construction, \( f_j(t) \) and \( Z_s^{f_j} \) are adapted to \( \mathcal{F}_t \) and are almost surely continuous. Thus, \( \{\theta_t\}_{t \geq 0} \) is predictable with respect to \( \{\mathcal{F}_t\}_{t \geq 0} \). Finally, note from the martingale property of stochastic integral (Theorem 6.2(3)), we have \( \mathbb{E}(M_t | \mathcal{F}_0) = \theta_0 + \int_0^t \theta_s \, dW_s = \theta_0 \). On the other hand, \( \mathbb{E}(M_t | \mathcal{F}_0) = M_0 \) by the definition of martingale. Thus \( \theta_0 = M_0 \), and (6.38) follows.

\[ \square \]

### 6.7 Fundamental Theorem of Asset Pricing

#### 6.7.1 Self-Financing Portfolio

Similar to the discrete time financial model, we focus on a simple situation that there are only two assets in the market: the bond \( B_t = e^{rt} \) and the stock \( S_t \). A trading strategy is denoted by a pair of predictable processes \( \{\psi_t, \phi_t\}_{t \geq 0} \) that indicates the amount of holding of each asset, i.e., investing \( \psi_t \) units of bond \( B_t \) and \( \phi_t \) units of stock \( S_t \) at time \( t \). This gives a portfolio \( V_t \) which worths

\[ V_t = \psi_t B_t + \phi_t S_t. \] (6.40)

Using the principle of no arbitrage pricing, we price a financial product by constructing a replicating portfolio. To be specific, suppose that we are at time \( t \) and we want to price a product with payoff \( C_T \) (a random variable) at time \( T \). We seek for a trading strategy \( \{\psi_t, \phi_t\}_{t \geq 0} \) such that the maturity value of the replicating portfolio \( V_T \) agrees with \( C_T \) in all cases (i.e., \( V_T = C_T \) a.s.). Then the price of the product \( C \) at \( t \) is given by \( V_t \).
It is important to note that, as the initial value \( V_0 \) of the replicating portfolio is the amount required to reproduce \( C_T \) at \( T \), the portfolio should be **self-financing**, i.e., *no external input/output* for \( V \) during \((0, T] \). This leads to the formula

\[
V_t = V_0 + \int_0^t \psi_u dB_u + \int_0^t \phi_u dS_u. \tag{6.41}
\]

To understand (6.41), note that if \( V \) is self-financing, then the changes in value of \( V \) is only due to the changes in the bond and stock values. Now, \( \psi_u dB_u + \phi_u dS_u \) is the infinitesimal change of the replicating portfolio at time \( u \), thus the right hand side of (6.41) is the sum over the infinitesimal changes over the time period \([0, t]\), which accumulates to the value of the portfolio at time \( t \), \( V_t \).

Equations (6.40) and (6.41) together motivate the following **self-financing** condition:

**Definition 6.5. (Self-financing Strategy)** A self-financing strategy is defined by a pair of adapted process \( \{ \psi_t \}_{t \geq 0}, \{ \phi_t \}_{t \geq 0} \), satisfying (6.40),

\[
\int_0^t |\psi_u| du + \int_0^t |\phi_u|^2 du < \infty, \text{ a.s.,} \tag{6.42}
\]

for all \( t > 0 \), and

\[
V_t = \psi_t B_t + \phi_t S_t = \psi_0 B_0 + \phi_0 S_0 + \int_0^t \psi_u dB_u + \int_0^t \phi_u dS_u, \text{ a.s.,} \tag{6.43}
\]

for all \( t > 0 \).

**Remark 6.2.** Equation (6.42) is required so that the integrals in (6.43) make sense. Note that the existence of \( \int_0^t \psi_u dB_u \) requires integrability of \( \psi_u \), while the the existence of \( \int_0^t \phi_u dS_u \) requires integrability of \( \phi_u^2 \).

We end this subsection with the following lemma that gives a representation of the discounted portfolio value by the discounted stock price. Such representation allows a simple way of pricing when equipped with the martingale measure to be introduced in the next subsection.

**Lemma 6.9 (Self-financing Strategy and Discounted Portfolio Value)** Let \( \{ \psi_t \}_{t \geq 0} \) and \( \{ \phi_t \}_{t \geq 0} \) be predictable processes satisfying (6.42) Let \( \tilde{V}_t = e^{-rt} V_t = e^{-rt} (\psi_t B_t + \phi_t S_t) \) and \( \tilde{S}_t = e^{-rt} S_t \), then \( \{ \psi_t, \phi_t \}_{t \geq 0} \) defines a self-financing strategy if and only if

\[
d\tilde{V}_t = \phi_t d\tilde{S}_t \text{ a.s.,}
\]

for all \( t \leq T \).

**Proof.** Using Ito’s Lemma, we have
\[ d\bar{V}_t = d\ e^{-rt}V_t \]
\[ = -re^{-rt}V_t \ dt + e^{-rt}dV_t \]
\[ = -re^{-rt}(\psi_tB_t + \phi_tS_t) \ dt + e^{-rt}dV_t \]
\[ = \phi_t e^{-rt}(-rS_t \ dt + dS_t) - e^{-rt}\phi_t dS_t - e^{-rt}\psi_t dB_t + e^{-rt}dV_t \]
\[ = \phi_t d\tilde{S}_t + e^{-rt}(dV_t - \psi_t dB_t - \phi_t dS_t), \]

since \( d\tilde{S}_t = d\ e^{-rt}S_t = e^{-rt}(-rS_t \ dt + dS_t). \) In other words, \( d\bar{V}_t = e^{-rt}d\tilde{S}_t \) if and only if \( dV_t - \psi_t dB_t - \phi_t dS_t = 0, \) i.e., \( \{\psi_t, \phi_t\}_{t \geq 0} \) is a self-financing strategy.

### 6.7.2 Self-Financing Replicating Portfolio and Martingale Measure

In Lemma 6.9, for a self-financing portfolio with trading strategy \( \{\psi_t, \phi_t\} \), we have obtained the representation \( d\tilde{V}_t = \phi_t d\tilde{S}_t \), or

\[ \tilde{V}_t = \tilde{V}_s + \int_s^t \phi_u d\tilde{S}_u, \quad (6.44) \]

for \( t \geq s \geq 0. \) If we only know the distribution of \( V_t \) but not the whole path \( \{\psi_t, \phi_t\}_{t \geq 0}, \) the following Lemma provides a way to find the value of a portfolio at time \( s, V_s, \) by simply taking expectation.

**Lemma 6.10 (Self-financing Strategy and Martingale Measure)** Let \( \tilde{V}_t(\psi, \phi) = e^{-rt}V_t(\psi, \phi) \) be a discounted value of a self-financing portfolio with some strategy (possibly unknown) \( \{\psi_t, \phi_t\}_{t \geq 0}. \) If there exists a probability measure \( Q \) such that the process \( \{\tilde{S}_u\} \) is a martingale under \( Q, \) then for any \( t \geq s \geq 0, \)

\[ V_s = E_Q(e^{-(t-s)}V_t|\mathcal{F}_s). \]

**Proof.** Using the argument in Theorem 6.2(c) and the fact that \( \tilde{S}_u \) is a martingale under \( Q, \) we have

\[
E_Q \left( \int_s^t \phi_u d\tilde{S}_u \bigg| \mathcal{F}_s \right) = \lim_{n \to \infty} E_Q \left( \sum_{j \geq s} \phi_j (\tilde{S}_{j+1} - \tilde{S}_j) \bigg| \mathcal{F}_s \right) \\
= \lim_{n \to \infty} E_Q \left( \sum_{j \geq s} \phi_j E_Q(\tilde{S}_{j+1} - \tilde{S}_j|\mathcal{F}_j) \bigg| \mathcal{F}_s \right) \\
= 0. \quad (\text{since } E_Q(\tilde{S}_{j+1} - \tilde{S}_j|\mathcal{F}_j) = 0)
\]

Thus the lemma is completed by taking conditional expectation \( E_Q(\cdot|\mathcal{F}_s) \) on both sides of (6.44).
Remark 6.3. The measure $Q$ is known as the **Martingale Measure** or the **Risk Neutral Probability Measure**. Under this measure, all discounted portfolio value is a martingale. The term **risk neutral** may be interpreted from the martingale property

$$E(e^{-rt}V_t | \mathcal{F}_s) = e^{-rs}V_s,$$

which asserts that all portfolio on average behaves the same as the bond $e^{rt}$.

### 6.7.3 Fundamental Theorem of Asset Pricing

Consider pricing a financial product with payoff $C_T$ at maturity. If we can find a self-financing replicating portfolio \( \{ \psi_t, \phi_t \}_{t \geq 0} \) such that $C_T = V_T$ a.s., then from the principle of no arbitrage, the price of the financial product at time $t$ is $V_t$. From Lemma 6.10, we have $V_t = E_Q(V_T | \mathcal{F}_t) = E_Q(C_T | \mathcal{F}_t)$, which gives the price of $C$ at time $t$. The Fundamental Theorem of Asset Pricing ensures that if $C_t$ is $\mathcal{F}_t$ measurable ($\{ \mathcal{F}_t \}$ is the natural filtration generated by $\{ S_t \}$), then the self-financing replicating portfolio $\{ \psi_t, \phi_t \}_{t \geq 0}$ such that $C_T = V_T$ a.s. exists. Thus it offers a very general way of pricing financial derivative.

**Theorem 6.11. (Fundamental Theorem of Asset Pricing)** Let $Q$ be the measure given by Lemma 6.10. Suppose that a financial derivative $C$ at time $T$ is given by the non-negative $\mathcal{F}_T$ measurable random variable $C_T$. If

$$E_Q(C_T^2) < \infty,$$

then $C$ is replicable and the value at time $t$ of any replicating portfolio is given by

$$V_t = E_Q\left( e^{-r(T-t)}C_T | \mathcal{F}_t \right). \quad (6.45)$$

**Proof.** Note that from Lemma 6.10, the price of any self-financing portfolio can be computed by taking expectation under the martingale measure. Thus it suffices to show that there exists a self-financing replicating portfolio $V$ with strategy $\{ \psi_t, \phi_t \}_{t \geq 0}$ such that $C_T = V_T$ a.s.. Let $V_t = e^{-rt}V_t$ and $\bar{C}_t = e^{-rt}C_t$. From Lemma 6.9, it suffice to find a $V_t$ such that $\bar{V}_T = \bar{C}_T$ a.s. and $\bar{V}_t$ has the representation

$$d\bar{V}_t = \phi_t d\bar{S}_t,$$

for some $\{ \bar{S}_t \}$ predictable process $\phi$.

The key to the proof is the clever construction $\bar{V}_t = E_Q(\bar{C}_T | \mathcal{F}_t)$, where $Q$ is the martingale measure given in Lemma 6.10 that makes $\bar{S}_t$ a martingale. This construction satisfies our requirement that 1) $\bar{V}_T = \bar{C}_T$ a.s., 2) $\bar{V}_t$ is self-financing. First, 1) is trivial since $\bar{C}_T$ is $\mathcal{F}_T$ measurable. Next, 2) follows from the fact that $\bar{V}_t$ is a martingale (see Example 4.5) and the Brownian Martingale Representation Theorem 6.8. To be specific, from Theorem 6.8, there exists a $\{ \mathcal{F}_t \}_{t \geq 0}$ predictable process $\{ \phi_t \}_{t \geq 0}$ such that with $Q$-probability one, $\bar{V}_t = \bar{V}_0 + \int_0^t \phi_s dW_s^Q$, i.e.,
\[ d\tilde{V}_t = \phi_t \, dW_t^Q, \quad (6.46) \]

where \( W_t^Q \) is a Brownian motion under \( Q \). Since \( \tilde{S}_t \) is a \( Q \) martingale by construction, the same argument can be repeated to yield \( d\tilde{S}_u = \theta_u \, dW_u^Q \) for some \( \theta_u \). Thus (6.46) can be expressed as \( d\tilde{V}_t = \frac{\phi_t}{\tilde{S}_t} \, d\tilde{S}_t \). Hence, Lemma 6.9 implies that \( V_t \) is a self-financing portfolio.

In summary, we have constructed a portfolio \( V_t = e^{rt} E_{Q}(\tilde{C}_T | \mathcal{F}_t) = E_{Q}(e^{-r(T-t)}C_T | \mathcal{F}_t) \) which is self-financing and satisfies \( V_T = C_T \) a.s. Thus, \( V_t \) gives the price of \( C \) at time \( t \).

Below is a summary of the general way of pricing in a continuous time financial model:

**Pricing via Risk Neutral Probability Measure**

To price a financial derivative \( C \) which has final payoff \( C_T \), first define a probability measure \( Q \) under which the discounted stock process \( \tilde{S}_t = e^{-rt}S_t \) takes the form \( d\tilde{S}_t = \theta_t \, dW_t^Q \), where \( W_t^Q \) is a Brownian motion under \( Q \). Then the price of \( C \) at time \( t \), \( V_t \), is obtained by

\[ V_t = E_Q(e^{-r(T-t)}C_T | \mathcal{F}_t). \]

**Remark 6.4.** In the Fundamental Theorem of Asset Pricing, \( C_t \) needs to be \( \mathcal{F}_t \) measurable, where \( \{ \mathcal{F}_t \}_{t \geq 0} \) is the natural filtration of the stock \( S_t \). Otherwise, the replication of \( C_t \) using \( S_t \) will not be possible. For example, if we want to price a European call option \( C \) on HSBC stock \( S^{(5)}_t \), we can form a replicating portfolio using the bond \( B_t \) and the stock \( S^{(5)}_t \), but not the bond and China mobile stock \( S^{(941)}_t \). In this case, \( C_t \) is \( \sigma(\{S^{(5)}_s\}_{s \leq t}) \) measurable but not \( \sigma(\{S^{(941)}_s\}_{s \leq t}) \) measurable.

One of the key ingredients of Fundamental Theorem of Asset Pricing is to find the martingale measure \( Q \) in Lemma 6.10. To achieve this, we need to assume a model for \( S_t \). In the next section, we describe the geometric Brownian motion model which allows \( Q \) to be found via the Girsanov Theorem.

### 6.8 Pricing Under Geometric Brownian Motion Model

#### 6.8.1 Geometric Brownian Motion (GBM) Model

The **Geometric Brownian Motion** model assumes that the stock price \( S_t \) follows an Ito process, i.e.,

\[ dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \quad (6.47) \]

where \( S_0 \) is the current price of the stock. Here, \( \mu \) is called the *drift* and \( \sigma \) is called the *volatility*. Both \( \mu \) and \( \sigma \) are assumed to be constants.
Applying Ito’s Lemma with $F(t,x) = \ln x$, we have

$$d \ln S_t = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dW_t,$$

or

$$S_t = S_0 \exp \left\{ \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\}, \tag{6.48}$$

Since $S_t$ is the exponential function of a Brownian motion, this motives the name Geometric Brownian Motion.

### 6.8.2 Martingale Measure for GBM

Recall that in the Fundamental Theorem of Asset Pricing we need to find a martingale measure $Q$ such that the discounted stock process $\tilde{S}_t = e^{-rt}S_t$ satisfies

$$d \tilde{S}_t = \theta_t dW_t^Q; \tag{6.49}$$

where $W_t^Q$ is a Brownian motion under $Q$ and $\{\theta_t\}_{t \geq 0}$ is a predictable process w.r.t. $\{\mathcal{F}_t\}_{t \geq 0}$. Now, under the GBM model (6.47), we have from Ito’s Lemma that

$$d \tilde{S}_t = (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dW_t$$

$$= \sigma \tilde{S}_t d \left( W_t + \frac{\mu - r}{\sigma} t \right).$$

Therefore, we need a measure $Q$ such that

$$W_t^Q \equiv W_t + \frac{\mu - r}{\sigma} t \tag{6.50}$$

is a B.M. under $Q$. The solution follows directly from the Girsanov Theorem 6.7 with $\theta_s = \frac{\mu - r}{\sigma}$, i.e., for $A \in \mathcal{F}_t$,

$$Q(A) = \int_A L_t dP, \tag{6.51}$$

where

$$L_t = e^{-\frac{\mu - r}{\sigma} t} - \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 t$$

is the Radon Nikodym Derivative.
6.8.3 Pricing In Action

Example 6.10. (European Call Option) An European call option offers a choice for the holder to buy a stock $S$ at the strike price $K$. Thus the payoff at maturity $T$ is

$$C_T = (S_T - K)^+,$$

where $X^+ = X \cdot I(X > 0)$. From (6.45), the price of this option at time 0 is given by

$$V_0 = E_Q(e^{-rT}(S_T - K)^+)$$

$$= E_Q(e^{-rT}(S_0e^{\left(-\frac{\sigma^2}{2}\right)T + \sigma W_T} - K)^+)$$

(from (6.48))

$$= E_Q(e^{-rT}(S_0e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W^Q_T} - K)^+)$$

(from (6.50))

$$= e^{-rT} \int_{-d_2\sqrt{T}}^{\infty} (S_0e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma x - K}) \phi(x,0,T) dx$$

(Change of variables: $u = \frac{x}{\sigma \sqrt{T}}$)

$$= \int_{-d_2}^{\infty} \left(S_0e^{-\frac{\sigma^2}{2}T + \sigma u \sqrt{T} - Ke^{-rT}}\phi(u)\right) du$$

$$= S_0 \Phi(d_1) - Ke^{-rT} \Phi(d_2),$$

where $\phi(x,0,T)$ is the p.d.f. of $N(0,T)$, $\Phi(x)$ is the p.d.f. and c.d.f. of $N(0,1)$ distribution, and $d_1 = d_2 + \sigma \sqrt{T}$.

Example 6.11. (Digital Options) A digital call option, also known as binary call option or a cash or nothing call option, has payoff structure

$$C_T = 1_{\{S_T \geq K\}}.$$

From (6.45), the price of this option at time 0 is given by
6.8 Pricing Under Geometric Brownian Motion Model

\[ V_0 = \mathbb{E}_Q \left( e^{-rT} 1_{\{S_T \geq K\}} \right) \]

\[ = \mathbb{E}_Q \left( e^{-rT} 1_{\{S_0 e^{\left( \mu - \frac{\sigma^2}{2}\right) T + \sigma W_T\} \geq K\}} \right) \quad \text{(from (6.48))} \]

\[ = \mathbb{E}_Q \left( e^{-rT} 1_{\{S_0 e^{\left( r - \frac{\sigma^2}{2}\right) T + \sigma W_T\} \geq K\}} \right) \quad \text{(from (6.50))} \]

\[ = e^{-rT} \int_{-d_2 \sqrt{T}}^{\infty} \phi(x,0,T) dx \quad \text{(Let } d_2 = \frac{\ln \frac{S_0}{K} + (r - \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \text{)} \]

\[ = e^{-rT} \int_{-d_2}^{\infty} \phi(x) dx = e^{-rT} \Phi(d_2), \]

where \( \phi(x,0,T) \), \( \phi(x) \) and \( \Phi(x) \) are defined in Example 6.10.

**Example 6.12. (Barrier Options)** A barrier option can be **activated** or **deactivated** if the asset price crosses a barrier before the maturity. By considering the combinations of the following two features, there are four types of barrier options:

i) Direction of hitting barrier:
   - **Up**: the barrier is hit from below; **Down**: the barrier is hit from above.

ii) Action when barrier is hit:
   - **In**: the option is activated; **Out**: the option is deactivated.

Note that these two features can be applied to any options. For examples, we have down-and-in European call options, up-and-out digital put options, or down-and-out Asian call options, etc...

Since the maximum or minimum of the stock price process determines whether the barrier is hit, its distribution will be useful in pricing barrier options. Recall from Lemma 5.10 the joint distributions of \( W_t \) and \( M_t = \max_{s \geq 0} W_s \), where \( W_t \) is a \( \mathbb{P} \)-B.M.:

\[ \mathbb{P}(M_t > a, W_t \in dx) = \phi \left( \frac{2a - x}{\sqrt{t}} \right) \frac{1}{\sqrt{t}} \]  \hspace{1cm} (6.52)

\[ \mathbb{P}(M_t \in da, W_t \in dx) = \phi \left( \frac{2a - x}{\sqrt{t}} \right) \frac{2(2a - x)}{t^{3/2}}, \] \hspace{1cm} (6.53)

for \( a \geq x \), where (6.53) is obtained from (6.52) by differentiation w.r.t. \( a \). Note that \( \mathbb{P}(M_t \in da, W_t \in dx) = 0 \) for \( a < x \) since the running maximum is never less than the current value. Similarly, we can see that

\[ \mathbb{P}(M_t > a, W_t \in dx) = \mathbb{P}(M_t > x, W_t \in dx) \]

for \( a < x \). We illustrate the pricing procedure using the following **up-and-in digital option** with payoff:
where $M_T \equiv \max_{t \geq 0} S_t$ and $K > S_0$. Again, from (6.45), the price of this option at time 0 is given by

$$V_0 = E_Q \left( e^{-rT} 1_{\{M_T > K\}} \right)$$

$$= E_Q \left( e^{-rT} 1_{\{ \max_{t \geq 0} S_t e^{\left( \frac{\mu - \frac{1}{2} \sigma^2}{\sqrt{T}} \right) t + \sigma W_t^Q} \geq K \} \right) \quad \text{(from (6.48))}$$

$$= E_Q \left( e^{-rT} 1_{\{ \max_{t \geq 0} S_t e^{\left( \frac{\mu - \frac{1}{2} \sigma^2}{\sqrt{T}} \right) t + \sigma W_t^Q} \geq K \} \right) \quad \text{(from (6.50))}$$

$$= E_Q \left( e^{-rT} 1_{\{ \max_{t \geq 0} S_t e^{\left( \frac{\mu - \frac{1}{2} \sigma^2}{\sqrt{T}} \right) t + \sigma W_t^Q} \geq K \} \right) \quad \text{(Girsanov: } W_t^Q = W_t^Q + bT \text{ is a } Q-B.M., b = \frac{r - \frac{\sigma^2}{2}}{\sigma})$$

$$= \int \int e^{-rT} 1_{\{ S_0 e^{\left( \frac{\mu - \frac{1}{2} \sigma^2}{\sqrt{T}} \right) t + \sigma W_t^Q} > K \} \} e^{hx - \frac{1}{2} b^2 T} \tilde{Q}(M_T \in da, W_T^Q \in dx) da dx \quad \text{(from (6.53), where } \tilde{M}_T = \max_{t \geq 0} W_T^Q \text{)}$$

$$= \int_{-\infty}^{\infty} e^{-rT} e^{hx - \frac{1}{2} b^2 T} \tilde{Q}(M_T > c \vee x, W_T^Q \in dx) dx \quad \text{(evaluate the inner integral)}$$

$$= \int_{-\infty}^{c} e^{-rT} e^{hx - \frac{1}{2} b^2 T} \phi \left( \frac{2c - x}{\sqrt{T}} \right) \frac{1}{\sqrt{T}} dx + \int_{c}^{\infty} e^{-rT} e^{hx - \frac{1}{2} b^2 T} \phi \left( \frac{x}{\sqrt{T}} \right) \frac{1}{\sqrt{T}} dx$$

$$= \int_{-\infty}^{c} e^{-rT} e^{hx - \frac{1}{2} b^2 T} e^{-\frac{(x - c)^2}{2T}} dx + \int_{c}^{\infty} e^{-rT} e^{hx - \frac{1}{2} b^2 T} e^{-\frac{x^2}{2T}} dx \quad \text{(Normal pdf: } \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2})$$

$$= e^{2bc - rT} \int_{-\infty}^{c} \frac{1}{\sqrt{2\pi T}} e^{-\frac{(x - c - bT)^2}{2T}} dx + e^{-rT} \int_{c}^{\infty} \frac{1}{\sqrt{2\pi T}} e^{-\frac{(x - bT)^2}{2T}} dx$$

$$= e^{2bc - rT} \left( \frac{c - bT}{\sqrt{T}} + e^{-rT} \left( 1 - \Phi \left( \frac{c - bT}{\sqrt{T}} \right) \right) \right).$$

**Example 6.13. (American Options)** The key feature of American options is that the time of exercising the option is not determined. Therefore, the “Maturity” has to be modeled by a stopping time $\tau$. The probabilistic properties about the stopping time developed in Chapter 5 will be useful in pricing American option.

Consider the American option where $S$ reaches $a, a > S_0$. Let $T_0 = \inf \{ t \geq 0 : S_t = a \} = \inf \{ t \geq 0 : S_0 e^{(r - \frac{1}{2} \sigma^2) t + \sigma W_t^Q} = a \}$, the payoff is given by
From (6.45), the price of this option at time 0 is given by

\[ V_0 = E_Q \left( K e^{-r T_n} \right) \]

\[ = E_Q \left( K e^{-r \inf \{ t \geq 0 : S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma W_t^Q} = a \} } \right) \]

\[ = E_Q \left( K e^{-r \inf \{ t \geq 0 : W_t^Q = c - bt \} } \right) \]

(set \( c = \frac{1}{\sigma} \log \frac{a}{S_0}, \ b = \frac{r - \sigma}{2} \))

\[ = Ke^{-(b + \sqrt{2r + b^2})}. \]  

(Example 6.9)

### 6.9 Exercises

**Exercise 6.1.** Let \( 0 = t_0^n < t_1^n < \cdots < t_n^n = T \), where \( t_j^n = \frac{jT}{n} \), be a partition of the interval \([0, T]\) into \( n \) equal parts. Using the identity \( a(b - a) = \frac{1}{2} (b^2 - a^2) - \frac{1}{2} (a - b)^2 \) and Theorem 5.14, show that

\[ \lim_{n \to \infty} \frac{1}{n-1} \sum_{j=0}^{n-1} W(t_j^n) (W(t_{j+1}^n) - W(t_j^n)) = \frac{1}{2} W_T^2 - T. \]  

(6.54)

Using the identity \( b(b - a) = \frac{1}{2} (b^2 - a^2) + \frac{1}{2} (a - b)^2 \) and Theorem 5.14, show that

\[ \lim_{n \to \infty} \sum_{j=0}^{n-1} W(t_j^n) (W(t_{j+1}^n) - W(t_j^n)) = \frac{1}{2} W_T^2 + T. \]  

(6.55)

Find the limit

\[ \lim_{n \to \infty} \sum_{j=0}^{n-1} W(p t_{j+1}^n + (1 - p) t_j^n) (W(t_{j+1}^n) - W(t_j^n)) , \]  

(6.56)

for \( p \in (0, 1) \).

**Exercise 6.2.** Show that if a sequence converges to some limit, then every subsequence converges to the same limit.

**Exercise 6.3.** Show that \( W^2 \) belongs to \( \mathcal{M}^2 \), where \( W_t \) is a Wiener process.

**Exercise 6.4.** Suppose that \( \tau \) is a stopping time w.r.t. the filtration \( \{ \mathcal{F}_t \}_{t \geq 0} \). Is there any \( t \) such that \( \tau \in \mathcal{F}_t \)?

**Exercise 6.5.** Using the notations in Theorem 6.7, show that if \( A \in \mathcal{F}_t \), then \( Q(A) := \int_A L_t(d\omega) \mathbb{P}(d\omega) = \int_A L_T(d\omega) \mathbb{P}(d\omega) \) for any \( T > t \). If \( s < t \), is \( Q(A) := \int_A L_s(d\omega) \mathbb{P}(d\omega) = \int_A L_t(d\omega) \mathbb{P}(d\omega) \) true? Prove it or give a counter example.
Exercise 6.6. Verify the equalities
\[ \int_0^T t \, dW_t = T W_T - \int_0^T W_t \, dt, \]  
\[ \int_0^T W_t^2 \, dt = \frac{1}{3} W_T^3 - \int_0^T W_t \, dt. \]  
\[ \text{(6.57)} \]
\[ \text{(6.58)} \]
and
Note that the integral on the right-hand side is a Riemann integral defined path-wise, i.e. defined separately for each \( \omega \in \Omega \).

Exercise 6.7. Suppose that \( g(x) \) is a continuous, and with \( t^n_i = iT/n, 0 = t_0 < t_1 < \cdots < t_n = T \) is a partition of \([0, T]\). Show that \( \lim_{n \to \infty} \sup_{t_i \leq t \leq t_{i+1}} |g(t^n_i) - g(t)| \to 0 \). If \( T = \infty \), does the convergence hold? Prove it or give a counter example.

**Exercise 6.8. (General Case of Ito’s Lemma)** Suppose that \( \psi_n(x) \) is a smooth function satisfying \( \psi_n(x) = 1 \) for any \( x \in [-n, n] \) and \( \psi_n(x) = 0 \) for \( x \notin [-n - 1, n + 1] \). Let \( F_n(t, x) = \psi_n(x) F(t, x) \), show that \( F_n(t, x) \) satisfies the conditions of Theorem 6.3 and has bounded partial derivatives \( (F_n)_x \) and \( (F_n)_x \) for each \( n \).

Since \( F(t, x) = F_n(t, x) \) for every \( t \in [0, T] \) and \( x \in [-n, n] \), the Ito’s Lemma holds for \( F \) on the set \( A_n = \{ \sup_{t \in [0, T]} |W_t| < n \} \). By show that \( \lim_{n \to \infty} P(A_n) = 1 \), deduce that Ito’s Lemma holds for \( F \) almost surely.

Exercise 6.9. For \( F(t, x) = x^3 \) we have \( F_t(t, x) = 0, F_x(t, x) = 3x^2 \) and \( F_{xx}(t, x) = 6x \). Verify that \( 3W_t^2 \in \mathcal{M}_2^2 \) and use Ito’s Lemma to show the equality
\[ d \left( W_t^2 \right) = 3W_t \, dt + 3W_t^2 \, dW_t. \]

Exercise 6.10. Show that the exponential martingale \( X_t = \exp \left\{ W_t - \frac{t}{2} \right\} \) is satisfies
\[ dX_t = X_t \, dW_t. \]

Exercise 6.11. Show that the initial value problem
\[ \begin{cases} 
  dX_t = aX_t \, dt + bX_t \, dW_t, \\
  X(0) = x_0
\end{cases} \]
has a solution given by \( X_t = x_0 \exp \left\{ \left( a - \frac{b^2}{2} \right) t + bW_t \right\} \).


Exercise 6.13. Show that for any two random variables \( X \) and \( Y \), \( \|X - Y\|_{L^2} = 0 \) implies \( X = Y \) a.s..

Consider a sequence of random variables \( X_n \). If \( \|X_n - X\|_{L^2} \to 0 \), does it imply \( X_n \to X \) a.s.? Prove or give counter examples.

Exercise 6.14. Use Ito’s Lemma to compute \( \text{E}(W_t^6) \).

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Exercise 6.15. Let $X$ and $\{X_n(\omega)\}_{n \geq 1}$ be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Recall the three mode of convergence of a sequence of random variables $\{X_n\}$ to $X$.

- (Almost Surely, a.s.) $\mathbb{P}(\omega : \lim_{n \to \infty} |X_n(\omega) - X(\omega)| = 0) = 1$.
- (Probability:) For each $\varepsilon > 0$, $\lim_{n \to \infty} \mathbb{P}(|X_n(\omega) - X(\omega)| > \varepsilon) = 0$.
- ($L^2$) $\lim_{n \to \infty} \mathbb{E}(|X_n - X|^2) = 0$.

Show that

1. By rewriting the limit notation as union/intersect operation, show that convergent a.s. implies convergent in probability.
2. By considering the inequality $1_{\{Y > a\}} \leq \frac{Y^2}{a^2} 1_{\{Y > a\}} \leq \frac{Y^2}{a^2}$, show that convergent in $L^2$ implies convergent in probability.
3. Find an example where $X_n$ converges to $X$ a.s. but not in $L^2$. Hints: $X_n(\omega)$ takes non-zero values in a decreasing portion of $\Omega$, but the value is huge when non-zero.
4. Find an example where $X_n$ converges to $X$ in $L^2$ but not a.s. Hints: $X_n(\omega)$ takes non-zero values in a decreasing portion of $\Omega$. The value is bounded when $X_n(\omega)$ is non-zero. But the subset of $\Omega$ where $X_n(\omega)$ is non-zero is “moving” around so that for each $\omega$ there are infinite many $n$ such that $X_n(\omega)$ is non-zero.

Exercise 6.16. Use the notation in Exercise 6.15. Recall the First Borel-Cantelli Lemma that $\mathbb{P}(\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} A_j) = 0$ if $\sum_{i=1}^{\infty} \mathbb{P}(A_i) < \infty$.

1. Let $A_j = \{|X_j - X| > 2^{-j}\}$. Show that $\mathbb{P}(\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} A_j) = 0$ implies $X_j \xrightarrow{a.s.} X$.
2. Suppose that $X_n \xrightarrow{P} X$. Show that there is a subsequence $X_{n_j}$ of $X_n$ such that $\mathbb{P}(|X_{n_j} - X| > 2^{-j}) < 2^{-j}$ along the subsequence.
3. Show that, if $X_n$ converges to $X$ in probability, then there is some subsequence such that $X_{n_j}$ converges to $X$ a.s..

Exercise 6.17. Find the quadratic variation of $X_t$ where $X_t$ is an Ito’s process satisfying (6.22). Show that $d[X]_t = b^2(t)dt$.

Exercise 6.18. Given $X_t = x_0 e^{at} + bW_t$, where $a, b \in \mathbb{R}$, verify that $X_t \in \mathcal{M}^2$.

Exercise 6.19. Let $S_t$ follow the GBM model (6.47), prove that

1. The density function $f(x)$ of $\frac{S_t}{S_0}$ is given by

   $$f(x) = \frac{1}{\sqrt{2\pi \sigma^2 t}} \exp \left\{ -\frac{\ln x - \left(\mu - \frac{\sigma^2}{2}\right)t}{2\sigma^2 t} \right\} 1_{\mathbb{R}^+}(x).$$

2. $E[S_t] = S_0 \exp \{\mu t\}$.
3. $\text{Var}(S_t) = S_0^2 \exp \{2\mu t\} \left(\exp \{\sigma^2 t\} - 1\right)$.

Exercise 6.20. Complete the calculation of $\mathbb{E}[\mathbb{I}_{\{\max_{s \geq 0} S_s > K\}}]$ in Example 6.8.
Exercise 6.21. Let \( \{X_t\} \) be a \( \{\mathcal{F}_t\} \) adapted process and \( \{L_t\} \) be the Radon-Nikodym process such that \( L_t = \frac{dQ}{dP} \bigg|_t \) is the R.N.D. of \( Q \) w.r.t. \( P \) when both are defined on \( (\Omega, \mathcal{F}_t) \). Let \( t > x \), show that

1. \( L_t \) is a \( P \) martingale.
2. For \( A \in \mathcal{F}_s \), \( \int_A X_t L_t \, dP = \int_A X_t L_s \, dP = \int_A X_s \, dQ \).

For \( A \in \mathcal{F}_s \), is \( \int_A X_t L_t \, dP = \int_A X_t L_s \, dP = \int_A X_s \, dQ \)?