Chapter 5
Brownian Motion

Brownian motion originated as a model proposed by Robert Brown in 1828 for the phenomenon of “continual swarming motion” of pollen grains suspended in water. In 1900, Bachelier proposed Brownian motion as a continuous time model of stock price fluctuations, which improves the discrete time models by allowing the stock price to change at any instant. The mathematical theory of Brownian motion was later developed by Norbert Wiener. Therefore, Brownian motion is also known as Wiener process.

5.1 Definition of Brownian motion

Brownian motion is a particular example of a continuous time stochastic process. To formally define a continuous time stochastic process \( \{X_t : t \geq 0\} \), it requires a probability triple \((\Omega, \mathcal{F}, P)\) such that \(X_t\) is \(\mathcal{F}\)-measurable for all \(t\). As in the discrete case, we shall rarely specify \((\Omega, \mathcal{F}, P)\) explicitly. On the other hand, the probability space \((\Omega_B, \mathcal{F}_B, P_B)\) induced by the Brownian motion can be defined as follows.

**Definition 5.1. (Probability Space of Brownian Motion)** The probability space induced by the Brownian motion, \((\Omega_B, \mathcal{F}_B, P_B)\), is given by

- \(\Omega_B = \{\text{all continuous functions } \omega : [0, \infty) \to \mathbb{R}\}\)
- \(\mathcal{F}_B\) is the \(\sigma\)-field generated by the finite dimensional sets, i.e., \(\mathcal{F}_B = \sigma\{\omega : \omega(t_i) \in B_i, i = 1, \ldots, n\}, n \in \mathbb{N}\), where \(B_i \in \mathcal{B}\) and \(t_i \in [0, \infty)\).
- \(P_B\) is a probability measure on \((\Omega_B, \mathcal{F}_B)\) such that \(P_B(\{\omega : \omega(0) = 0\}) = 1\) and

\[
P_B\{\omega : \omega(t_i) \in B_i, i = 1, \ldots, n\} = \int_{B_1} \cdots \int_{B_n} \phi_{t_1 \ldots t_n}(x_1, \ldots, x_n) \, dx_n \cdots dx_1 \tag{5.1}
\]

where \(\phi_{t_1 \ldots t_n}(x_1, x_2, \ldots, x_n)\) is the joint p.d.f. of a Gaussian random vector with zero mean and covariance matrix \((\text{min}(t_i, t_j))_{i,j}\).
With the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), the Brownian motions are defined as the elements in \(\Omega\) with the following properties. To simplify notations, we denote \((\Omega, \mathcal{F}, \mathbb{P})\) by \((\Omega, \mathcal{F}, \mathbb{P})\).

**Definition 5.2. (Brownian motion)** A real-valued stochastic process \(\{W_t: t \geq 0\}\) is a \(\mathbb{P}\)-Brownian motion (or a \(\mathbb{P}\)-Wiener process) if for some \(\sigma \in \mathbb{R}^+\), under \(\mathbb{P}\),

1. **(Stationary increment)** For each \(s \geq 0\) and \(t > 0\) the random variable \(W_{t+s} - W_t\) is normally distributed with mean zero and variance \(\sigma^2 t\).
2. **(Independent increment)** For each \(n \geq 1\) and any time indexes \(0 \leq t_0 < t_1 < \cdots < t_n\), the random variables \(\{W_{r} - W_{r-1}\}_{r=1,\ldots,n}\) are independent.
3. **(Starts at 0)** \(\mathbb{P}(W_0 = 0) = 1\).

**Remark 5.1.**

1. **(Standard Brownian Motion)** The process \(\{W_t: t \geq 0\}\) with \(\sigma^2 = 1\) is called standard Brownian motion. The parameter \(\sigma^2\) is known as the variance parameter. For simplicity, unless otherwise stated we shall assume that \(\sigma^2 = 1\).
2. Brownian motion started from \(x\) can be obtained as \(\{x + W_t: t \geq 0\}\).

**Example 5.1.** For any positive constant \(c\), the process \(\tilde{W}_t = c^{-1/2}W_{ct}\) is a Brownian motion. To see this, we verify the three conditions in Definition 5.2.

1. **(Stationary increment)** For each \(s \geq 0\) and \(t > 0\) the random variable
   \[
   \tilde{W}_{t+s} - \tilde{W}_s = c^{-1/2}(W_{ct(t+s)} - W_{ct}) \overset{D}{=} c^{-1/2}N(0, ct) = N(0, t).
   \]
2. **(Independent increment)** For each \(n \geq 1\) and any times \(0 = t_0 < t_1 < \cdots < t_n\), the random variables \(\{\tilde{W}_r - \tilde{W}_{r-1}\}_{r=1,\ldots,n}\) are independent since \(\{W_{ct_r} - W_{ct_{r-1}}\}_{r=1,\ldots,n}\) are independent.
3. **(Starts at 0)** \(\tilde{W}_0 = c^{-1/2}W_{ct0} = 0\) since \(W_0 = 0\).

**Remark 5.2. (Fractal)** The above example shows that Brownian motion is a fractal, i.e., a set of points that shows self-similar pattern: For example, if \(c = 0.01\), the graph of the rescaled process \(\tilde{W}_t = c^{-1/2}W_{ct}\) on \(t \in [0, 1]\) is the graph of 10 times of the original Brownian motion \(W_t\) from \(t \in [0, 0.01]\). However, the shapes of \(\{W_t\}_{t \in [0,1]}\) and \(\{\tilde{W}_t\}_{t \in [0,1]}\) are similar, since they are both standard Brownian motions. In other words, it doesn’t matter what scale you examine the Brownian motion, they look just the same.

**Example 5.2. (Translational Invariance)** For any \(s \geq 0\), the process \(\tilde{W}_t = W_{t+s} - W_s\) is a standard Brownian motion. The verification is left to Exercise 5.1.

Combining conditions 1 and 2 of Definition 5.2, elementary algebra yields the transition probabilities of standard Brownian motion.
Definition 5.3. (Transition Probabilities) The transition probability of a standard Brownian motion from \( x \) to \( y \) in time \( t \) is the conditional probability density of \( W_{t+s} \) given \( W_s = x \), which is given by
\[
p(t, x, y) := \frac{1}{\sqrt{2\pi t}} \exp \left\{ -\frac{(x-y)^2}{2t} \right\}.
\]

Transition probability is a useful tool in establishing many probabilistic properties of Brownian motion. For example, a simpler formula for the joint density can be deduced (compare to (5.1)).

Proposition 5.1. (Joint Probability Density) For \( 0 = t_0 < t_1 < t_2 < \cdots < t_n \), with \( x_0 = 0 \), the joint probability density function of \( W_{t_1}, \cdots, W_{t_n} \) is given explicitly as
\[
 f(x_1, \cdots, x_n) = \prod_{j=1}^{n} p(t_j - t_{j-1}, x_{j-1}, x_j).
\]
The joint distributions of \( W_{t_1}, \cdots, W_{t_n} \) for each \( n \geq 1 \) and all \( t_1, \cdots, t_n \) are known as the finite dimensional distributions of the process.

Proof. By simple transformation, the joint density of \( W_{t_1} = x_1, W_{t_2} = x_2, \ldots, W_{t_n} = x_n \) is equivalent to the joint density of \( W_{t_1} = x_1, W_{t_2} - W_{t_1} = x_2 - x_1, \ldots, W_{t_n} - W_{t_{n-1}} = x_n - x_{n-1} \). From the stationary and independent increment properties, the latter set of random variables are independent and have probability densities equal to the transition probabilities \( p(t_j - t_{j-1}, x_{j-1}, x_j) \), \( j = 1, \ldots, n \), respectively. By the independence, the joint density is given by the product of the transition probabilities and thus Proposition 5.1 follows. \( \square \)

The conditional mean and variance can be computed easily using the stationary and independent increment of Brownian motion (See Exercise 5.6).

Lemma 5.1 For any \( s, t > 0 \),
1. \( E(W_{t+s} - W_s | \{W_r : 0 \leq r \leq s\}) = 0 \),
2. \( \text{Cov}(W_s, W_t) = s \wedge t \).

5.2 Continuity and Non-Differentiability of Brownian Motion

In this section we show that Brownian motion is continuous but non-differentiable. In other words, Brownian motion has a rough path.

Theorem 5.2. (Properties of Brownian Motions)
1. Brownian motion \( W_t \) is continuous in probability at any \( t \), i.e., for any \( \varepsilon > 0 \),
\[
 \lim_{u \to 0} P(|W_{t+u} - W_t| \geq \varepsilon) = 0.
\]
2. Brownian motion $W_t$ is almost surely non-differentiable at any $t$.

Proof. 1. (Continuity). From the translation invariance property of Brownian motion established in Example 5.2, we have $W_{t+s} - W_s \overset{D}{=} W_t$ for any $s \geq 0$. Thus it suffices to show the continuity of a Brownian motion at $t = 0$. We need to show that for any $\varepsilon > 0$

$$\lim_{t \to 0} \mathbb{P}(|W_t| \geq \varepsilon) = 0,$$

which holds because

$$\mathbb{P}(|W_t| \geq \varepsilon) = 2 \left(1 - \Phi \left( \frac{\varepsilon}{\sqrt{t}} \right) \right) \to 2(1 - \Phi(\infty)) = 0,$$

as $t \to 0$, where $\Phi(\cdot)$ is the c.d.f. of a standard Normal distribution. In fact, Brownian motion is almost surely continuous, i.e., $\mathbb{P}(|W_{t+u} - W_t| = 0) = 1$, but the proof is much more involved.

2. (Non-differentiability). Again, by the translation invariance property, it suffices to focus on the case $t = 0$. If $W_t$ is differentiable at 0, then $W_t$ converges as $t \to 0$. We will show that, with probability 1, for any $n$, there exists a $t \in \left(0, \frac{1}{n^4}\right]$ such that $\frac{|W_t|}{t} > n$, which implies that $W_t$ is not differentiable at 0. Let

$$A_n = \left\{ \frac{|W_t|}{t} > n \text{ for some } t \in \left(0, \frac{1}{n^4}\right]\right\}.$$

Since $\left\{ \frac{|W_t|}{t} > n \text{ at } t = \frac{1}{n^4} \right\} \subseteq A_n$, we have

$$\mathbb{P}(A_n) \geq \mathbb{P} \left( \frac{|W_{1/n^4}|}{1/n^4} > n \right)$$

(\text{Put } t = \frac{1}{n^4})

$$= \mathbb{P} \left( n^2 W_{1/n^4} > \frac{1}{n} \right)$$

(Simple algebra)

$$= \mathbb{P} \left( |\tilde{W}_t| > \frac{1}{n} \right)$$

(Example 5.1: $\tilde{W}_t = n^2 W_{1/n^4}$ is a B.M.)

$$\to 1,$$

as $n \to \infty$. Using the fact that $A_n$ is a contracting sequence of events, we have

$$\mathbb{P} \left( W_t \text{ is non-differentiable at } 0 \right) \geq \mathbb{P} \left( \bigcap_{n=1}^{\infty} A_n \right) = \lim_{n \to \infty} \mathbb{P}(A_n) = 1.$$
Indeed, whether such a bizarre process actually exists is far from obvious. We will devote next section to show the existence of B.M. by explicitly constructing a stochastic process that satisfies Definition 5.2.

5.3 Lévy’s construction of Brownian motion

In this section we present a construction of B.M. due to Lévy’s polygonal approximation.

5.3.1 A polygonal approximation

Lévy’s idea is to construct a path of Brownian motion by a limit of polygonal interpolation. Particularly, by specifying the endpoints of the process and then infilling the values in the middle, we can ensure the almost sure convergence of the approximation. We require the following lemma.

Lemma 5.3 Suppose that \( \{W_t : t \geq 0\} \) is a Brownian motion starts at \( x_0 \). Conditional on \( W_t = x_1 \), the probability density function of \( W_{t/2} \) is

\[
p_{t/2}(x) := \sqrt{\frac{2}{\pi t}} \exp \left\{ -\frac{1}{2} \left( x - \frac{1}{2} (x_0 + x_1) \right)^2 \right\}.
\]

In other words, the \( W_{t/2} | \{W_0 = x_0, W_t = x_1\} \sim N(\frac{x_0 + x_1}{2}, \frac{t}{4}) \).

Proof. By the property of B.M., \( (W_{t/2}, W_t) \sim N((x_0, x_0), \Sigma) \), where \( \Sigma = \begin{pmatrix} t/2 & t/2 \\ t/2 & t \end{pmatrix} \).

Direct calculations show that \( |\Sigma| = t^2/4 \) and \( \Sigma^{-1} = \begin{pmatrix} 4/t & -2/t \\ -2/t & 2/t \end{pmatrix} \). Thus, the joint p.d.f. can be expressed as

\[
f_{W_{t/2}, W_t}(x, y) = \frac{1}{2\pi \sqrt{|\Sigma|}} e^{-\frac{1}{2} (x - x_0, y - x_0) \Sigma^{-1} (x - x_0, y - x_0)}
\]

\[
= \frac{1}{\sqrt{\pi t}} e^{-\frac{1}{2} ((x - x_0)^2 + (y - x)^2)}
\]

Since the marginal distribution of \( W_t \) is \( f_{W_t}(y) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-(y-x_0)^2}{2t}} \), the conditional distribution of \( W_{t/2} \) given \( W_t \) is

\[
f_{W_{t/2}|W_t=x_1}(x) = \frac{f_{W_{t/2}, W_t}(x, x_1)}{f_{W_t}(x_1)}
\]

\[
= \frac{1}{\sqrt{2\pi (t/4)}} e^{-\frac{1}{2(t/4)} \left( x - \frac{x_0 + x_1}{2} \right)^2}.
\]
completing the proof.

Without loss of generality we take the range of \( t \) to be \([0, 1]\). Lévy’s construction builds (inductively) a polygonal approximation to the Brownian motion from a countable collection of independent standard normal random variables. In particular, in the \( n \)-th step of the approximation, a process \( X_n = X_n(t) \) on \( t = [0, 1] \) will be defined on the grid \( G_n = \{ k/2^n \}_{k=0,1,...,2^n} \) using the i.i.d. Normal variables \( Z_k(k2^{-n}) \). As the grid \( G_n \) becomes finer as \( n \to \infty \), the B.M. is defined by the limit of \( X_n \). The construction is illustrated in Figure 5.1.

### Levy’s Construction of Brownian motion:

- **Initialization.** Set \( X_0(0) = 0, X_0(1) = Z(1) \) and
  
  \[ X_0(t) = tZ(1), \]

  i.e., \( X_0 \) is a linear function on \([0, 1]\) with vertices on the grid \( G_0 = \{ k/2^0 \}_{k=0,1} \).

- **The inductive Step.** Given \( X_n \) with vertices on the grid \( G_n \), we define \( X_{n+1} \) with vertices on \( G_{n+1} \) by infilling more points in the set \( G_{n+1} \backslash G_n \). In other words, \( X_{n+1} \) takes the same value as \( X_n \) in the grid \( G_n \). That is, denoting \( X_n^k \triangleq X_n(k/2^n) \) and \( X_{n+1}^k \triangleq X_{n+1}(k/2^{n+1}) \), we set

  \[ X_{n+1}^{2k} = X_n^k \quad \text{for} \quad k = 1, \ldots, 2^n. \tag{5.2} \]

  It remains to infill \( X_{n+1}^{2k-1} \) for \( k = 1, \ldots, 2^n \). Using Lemma 5.3 with \( (x_0, x_1) = (X_{n+1}^{2k-2}, X_{n+1}^{2k}) \) and \( t = 1/2^n \), the conditional distribution of \( X_{n+1}^{2k-1} \) is \( N(\frac{1}{2}(X_{n+1}^{2k-2} + X_{n+1}^{2k}), 2^{-n-2}) \). Therefore, we can generate \( X_{n+1}^{2k-1} \) by

  \[ X_{n+1}^{2k-1} = \frac{1}{2}(X_{n+1}^{2k-2} + X_{n+1}^{2k}) + 2^{-(n/2+1)}Z_{2k-1}. \tag{5.3} \]

  Thus, we obtain \( X_{n+1}(t) \) on the grid \( G_{n+1} \). Next, define \( X_{n+1}(t) \) on \( t \in [0, 1] \) by joining the points, i.e., for \( k = 0, \ldots, 2^{n+1} \),

  \[ X_{n+1}(t) = \frac{1}{2}(X_{n+1}^{2k} + X_{n+1}^{2k+1}) + (2^{n+1}t - k)(X_{n+1}^{2k+1} - X_{n+1}^{2k}) \quad \text{for} \quad t \in \left[ \frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}} \right]. \]

- **Repeat the induction forever** (\( n \to \infty \)).

### Remark 5.3.

The \( n \)-th process, \( X_n(t) \), is linear in each interval \([ (k-1)2^{-n}, k2^{-n} ] \), is continuous in \( t \) and satisfies \( X_n(0) = 0 \). It is thus determined by the values \( \{X_n^k : k = 1, \ldots, 2^n \} \). By (5.2) and the linear property in the construction, we have

\[ \frac{1}{2}(X_{n+1}^{2k-2} + X_{n+1}^{2k}) = \frac{1}{2}(X_n^{k-1} + X_n^k) = X_n((2k-1)/2^{n+1}). \]

Thus (5.3) can be written as
5.3 Lévy’s construction of Brownian motion

Fig. 5.1 Lévy’s sequence of polygonal approximations to Brownian motion.

\[ X_{n+1}^{2k-1} = X_n \left( \frac{(2k-1)/2^n+1}{2} \right) + 2^{-\frac{n}{2}+1} Z_{n+1}^{2k-1}. \]  

The representation (5.4) will be useful in the next subsection.

5.3.2 Convergence to Brownian motion

To justify the above construction of B.M., it is important to show the existence of \( \lim_{n \to \infty} X_n(t) \). Since \( X_n(t) \) is a random function, the limit should be understood as a random function \( X(t) \) on \( (\Omega_B, \mathcal{F}_B) \). We will show that \( X_n \) converges uniformly to \( X \) almost surely, i.e.,

\[ P( \lim_{n \to \infty} \max_{t \in [0,1]} (X_n(t) - X(t)) = 0 ) = 1. \]  

The representation (5.4) will be useful in the next subsection.

Note that the term “uniform” refers to the operation \( \max_{t \in [0,1]} \). Moreover, we need to verify that the limit \( X(t) \) satisfies Definition 5.2, hence a Brownian motion. This is the content of the following theorem.

**Theorem 5.4. (Convergence of the Polygonal Construction of Brownian Motion)**

1. There exists a random function \( X(t) \) on \( (\Omega_B, \mathcal{F}_B) \) satisfying (5.5).
2. The \( X(t) \) satisfies Definition 5.2.

**Proof.**

1. (Existence of Limit). Notice that \( \max_{t \in [0,1]} |X_{n+1}(t) - X_n(t)| \) is attained at the vertex \( t \in \left\{ (2k-1)2^{-n+1} : k = 1, 2, \ldots, 2^n \right\} \). Using (5.4), we have
\[ P \left\{ \max_{t \in [0,1]} |X_{n+1}(t) - X_n(t)| \geq 2^{-\frac{3}{4}} \right\} \]
\[ = P \left\{ \max_{1 \leq k \leq 2^n} Z \left( (2k-1)2^{-(n+1)} \right) \geq 2^{\frac{5}{2}} - ^1 \right\} \]
\[ \leq 2^n P \left\{ Z(1) \geq 2^{\frac{5}{2}} \right\} . \]

Now using the result of Exercise 5.3 (with \( t = 1 \)), for \( x > 0 \)
\[ P \{ Z(1) \geq x \} \leq \frac{1}{x \sqrt{2\pi}} \exp \left\{ -\frac{x^2}{2} \right\}, \]
and combining this with the fact that
\[ \exp \left\{ -2^{\frac{5}{2}} + 1 \right\} < 2^{-2n+2}, \]
we obtain that for \( n \geq 4 \)
\[ 2^n P \left\{ Z(1) \geq 2^{\frac{5}{2}} + 1 \right\} \leq \frac{2^n}{2^{\frac{5}{2}} + 1} \frac{1}{\sqrt{2\pi}} \exp \left\{ -2^{\frac{5}{2}} + 1 \right\} \]
\[ \leq \frac{2^n}{2^{\frac{5}{2}} + 1} 2^{-2n+2} < 2^{-n}. \]

Consider now for \( k > n \geq 4 \)
\[ P \left\{ \max_{t \in [0,1]} |X_k(t) - X_n(t)| \leq 2^{-\frac{3}{4}} + 3 \right\} \]
\[ \geq P \left\{ \sum_{j=n}^{k-1} \max_{t \in [0,1]} |X_{j+1}(t) - X_j(t)| \leq 2^{-\frac{3}{4}} + 3 \right\} \]
\[ \geq P \left\{ \max_{t \in [0,1]} |X_{j+1}(t) - X_j(t)| \leq 2^{-\frac{3}{4}}, j = n, \ldots, k - 1 \right\} \]
\[ \geq 1 - \sum_{j=n}^{k-1} 2^{-j} \geq 1 - 2^{-n+1}, \]
where the second inequality follows from the fact that if \( m_j \leq 2^{-j/4} \) for all \( j = n, \ldots, k - 1 \), then \( \sum_{j=n}^{k-1} m_j \leq \sum_{j=n}^{k-1} 2^{-j/4} \leq 2^{-n/4+3} \). In other words, we have
\[ P \left\{ \max_{t \in [0,1]} |X_k(t) - X_n(t)| \geq 2^{-\frac{5}{4}} + 3 \right\} \leq 2^{-n+1} \]
for all \( k > n \) (The case \( k = n \) is trivial). Since the maximum of \( X_k(t) \) can only increase by the addition of a new vertex, the event on the left is increasing with \( k \), hence
\[ \mathbb{P} \left( \bigcup_{k \geq n} \left\{ \max_{t \in [0,1]} |X_k(t) - X_n(t)| \geq 2^{-n^4 + 3}\right\} \right) \]
\[ = \lim_{k \to \infty} \mathbb{P} \left( \max_{t \in [0,1]} |X_k(t) - X_n(t)| \geq 2^{-n^4 + 3}\right) \leq 2^{-n^4 + 1}. \]

In particular, for any \( \varepsilon > 0 \),
\[ \lim_{n \to \infty} \mathbb{P} \left( \bigcup_{k \geq n} \left\{ \max_{t \in [0,1]} |X_k(t) - X_n(t)| \geq \varepsilon \right\} \right) = 0, \]
which implies the existence of the limit. Denote the limit by \( X(t) = \lim_{n \to \infty} X_n(t) \).

2. **(The Limit is B.M.)** In the first step of the construction, \( X_0(0) = 0 \) and \( X_0(1) = Z(1) \) have the same distributional property of B.M., i.e., \( W_0 = 0 \) and \( W_1 \sim N(0, 1) \). In each induction step, a new vertex is constructed using the covariance structure of the B.M.. Thus Properties 1-3 in Definition 5.2 are automatically satisfied for \( X_n(t) \) restricted to the grid \( G_n = \{ k2^{-n} \}_{k=0,1,\ldots,2^n} \). Since we don’t change the values of \( X_k(t) \) on \( t \in G_n \) for \( k > n \), the same must be true for the limit \( X \) on \( \bigcup_{n=1}^{\infty} G_n \). Before we show Properties 1-3 on the whole interval \([0,1] \), we show the continuity of \( X(t) \): For any \( \varepsilon > 0 \) and \( n \geq 0 \),
\[ \mathbb{P}(|X(t + \delta) - X(t)| > \varepsilon) \leq \mathbb{P}(|X(t + \delta) - X_n(t + \delta)| > \varepsilon/3) + \mathbb{P}(|X_n(t + \delta) - X_n(t)| > \varepsilon/3) + \mathbb{P}(|X_n(t) - X(t)| > \varepsilon/3). \]
Note that the middle term goes to zero as \( \delta \to 0 \) by the continuity of \( X_n \). Also, from (5.5) the first and the third term can be arbitrarily small because \( n \) can be arbitrarily large. Thus \( \mathbb{P}(|X(t + \delta) - X(t)| > \varepsilon) \) converges to 0 as \( \delta \to 0 \), yielding the continuity of the limit.
Finally, note that any real number on \([0,1] \) can be arbitrarily closely approximated by some elements in \( \bigcup_{n=1}^{\infty} G_n \). Thus by approximating any \( 0 < t_1 < t_2 < \cdots < t_n < 1 \) from \( \bigcup_{n=1}^{\infty} G_n \), the continuity of \( X(t) \) implies that all Properties hold without restriction for \( t \in [0,1] \).

### 5.4 The Reflection Principle and Hitting Times

Having proved that Brownian motion actually exists, we now explore more properties related to B.M.
5.4.1 Continuous time martingale and Stopping times

We have introduced martingale and stopping time in the discrete time context. In fact, similar notions can be defined in continuous time. In this subsection we define continuous time martingale and stopping time and state some useful results.

**Definition 5.4. (Continuous Time Filtration)** The sequence of $\sigma$-field $\{\mathcal{F}_t\}_{t \geq 0}$ is called a filtration if $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \leq t$.

**Definition 5.5. (Continuous Time Stochastic Process)** A continuous time stochastic process is a random function $\{X_t\}_{t \geq 0}$ (or $\{X(t)\}_{t \geq 0}$, a function of $t$), from some abstract measurable space $(\Omega, \mathcal{F})$ to the measurable space $(\mathcal{C}, \sigma(\mathcal{C}))$, where $\mathcal{C}$ is some space of functions. Some commonly used space of function includes $\mathcal{C}^k$, the space of $k$-th time differentiable functions, and $\mathcal{D}$, the space of right continuous functions with left limits.

**Definition 5.6. (Natural Filtration)** If the stochastic process $\{X_t\}$ satisfies $X_t \in \mathcal{F}_t$ for all $t \geq 0$, then $\{X_t\}$ is adapted to $\{\mathcal{F}_t\}$. If $\mathcal{F}_X = \sigma\{X_s, s \leq t\}$, then the stochastic process $\{X_t\}$ is adapted to $\mathcal{F}_X$ by construction. Hence, $\mathcal{F}_X$ is called the natural filtration of $X_t$.

**Example 5.3.**

i) If $Z_t = \int_0^t X_s \, ds$, then $\{Z_t\}$ is adapted to $\mathcal{F}_X$.

ii) If $M_t = \max_{0 \leq s \leq t} W_s$, then $\{M_t\}$ is adapted to $\mathcal{F}_W$.

iii) If $Z_t = W_{t+1}^2 - W_t^2$, then $\{Z_t\}$ is NOT adapted to $\mathcal{F}_W$.

**Definition 5.7. (Predictable Process)** Given a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, the stochastic process $\{X_t\}_{t \geq 0}$ is predictable if $X_t$ is $\mathcal{F}_t$-measurable, where

$$\mathcal{F}_t = \bigcup_{s < t} \mathcal{F}_s.$$ 

**Definition 5.8. (Continuous Time Martingale)** Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\{\mathcal{F}_t\}_{t \geq 0}$. A stochastic process $\{M_t\}$ is called a $(P, \{\mathcal{F}_t\})$ martingale if

i) $\{M_t\}$ is adapted to $\{\mathcal{F}_t\}$.

ii) $E(|M_t|) < \infty$.

iii) $E(M_t | \mathcal{F}_s) = M_s$ if $s < t$.

**Example 5.4.** Let $\{W_t\}$ be a standard Brownian motion and $\{\mathcal{F}_t^W\}$ be the natural filtration of the B.M., then

i) $W_t$ is a $(P, \{\mathcal{F}_t^W\})$ martingale.

ii) $W_t^2 - t$ is a $(P, \{\mathcal{F}_t^W\})$ martingale.

**Definition 5.9. (Stopping Time)** Given a stochastic process $\{X_t\}$ adapted to a filtration $\{\mathcal{F}_t\}$, a random variable $\tau$ is called a stopping time if

$$\{\tau \leq t\} \in \mathcal{F}_t.$$
In other words, with information $\mathcal{F}_t$ up till time $t$, we can determine whether or not $\tau \leq t$.

**Example 5.5.** Given a standard Brownian motion $W_t$, consider the first hitting time $T_a = \inf\{t \geq 0 : W_t = a\}$. It is easy to see that $T_a$ is a stopping time because

$$\{T_a \leq t\} = \{\exists s \in [0,t] : W_s = a\} \in \mathcal{F}_t,$$

(the set in the middle depends only on $\{W_s : 0 \leq s \leq t\}$). Notice also that, by definition, if $T_a < \infty$, then $W_{T_a} = a$.

Other examples of stopping times include $T_{|a|} := \inf\{t \geq 0 : |W_t| = a\}$ and $T_{(a,b)} := \inf\{t \geq 0 : |W_t| \notin (a,b)\}$. Just as for random walks, an example of a random time that is not a stopping time is the last time that a process hits some level. \[\Box\]

From the property of independent increment, Brownian motion has no memory. That is, if $\{W_t : t \geq 0\}$ is a Brownian motion and $s \geq 0$ is any fixed time, then $\{W_t + s - W_s : t \geq 0\}$ is also a Brownian motion, independent of $\{W_r : 0 \leq r \leq s\}$. In fact, we have the following non-trivial property of Brownian motion (the proof is out of scope).

**Theorem 5.5. (Brownian motion after stopping)** If $W_t$ is a Brownian motion and $\tau$ is a stopping time adapted to $\mathcal{F}_t$, then

$$\tilde{W}_t = W_t + \tau - W_\tau,$$

for $t \geq 0$ is also a Brownian motion.

Similar to the discrete time case, we have the following theorems about continuous time martingale and stopping time.

**Theorem 5.6. (Stopped Martingale is a Martingale)** If $X_t$ is a $\{\mathbb{P}, \{\mathcal{F}_t\}\}$ martingale and $\tau$ is a stopping time adapted to $\{\mathcal{F}_t\}$, then $X_{\tau \wedge t}$ is also a $\{\mathbb{P}, \{\mathcal{F}_t\}\}$ martingale.

**Theorem 5.7. (Optional Stopping Theorem)** Suppose that $X_t$ is a $\{\mathbb{P}, \{\mathcal{F}_t\}\}$ martingale and $\tau$ is a stopping time adapted to $\{\mathcal{F}_t\}$ such that the following conditions hold:

1. $\tau < \infty$ a.s.
2. $X_\tau$ is integrable, i.e., $E(|X_\tau|) < \infty$.
3. $E(X_t 1_{\{\tau > t\}}) \to 0$ as $t \to \infty$.

Then

$$E(X_\tau) = E(X_0).$$
5.4.2 The reflection principle

Not surprisingly, there is often much to be gained from exploiting the symmetry inherent in Brownian motion. As a warm-up we calculate the distribution function of \( T_a = \min\{t : W_t = a\} \).

Lemma 5.8 (Distribution function of first hitting time) Let \( \{W_t : t \geq 0\} \) be a \( \mathbb{P} \)-Brownian motion started from \( W_0 = 0 \) and let \( a > 0 \), then

\[
P\{T_a < t\} = 2P\{W_t > a\}.
\]

Proof. If \( W_t > a \), then by continuity of the Brownian path, \( T_a < t \). Moreover, from Theorem 5.5, \( \{W_t + T_a - W_{T_a} : t \geq 0\} \) is a Brownian motion. So, by symmetry,

\[
P\{W_t - W_{T_a} > 0 | T_a < t\} = \frac{1}{2}.
\]

Thus

\[
P\{W_t > a\} = P\{\{T_a < t\} \cap \{W_t - W_{T_a} > 0\}\}
= P\{T_a < t\} P\{W_t - W_{T_a} > 0 | T_a < t\}
= \frac{1}{2} P\{T_a < t\}.
\]

By verifying the definition of Brownian motions, we have a more refined version of the above idea.

Lemma 5.9 (The reflection principle) Let \( \{W_t : t \geq 0\} \) be a standard Brownian motion and let \( T \) be a stopping time. Define

\[
\tilde{W}_t := \begin{cases} W_t, & \text{if } t \leq T; \\ 2W_T - W_t, & \text{if } t > T. \end{cases}
\]

then \( \\{\tilde{W}_t : t \geq 0\} \) is also a standard Brownian motion.

Notice that if \( T = T_a \) (the first hitting time on \( a \)), then \( \tilde{W}_t \) reflects the portion of the path after \( T_a \) about the line \( x = a \) (see Figure 5.2). The reflection principle is the key to prove the following result, which is useful to price certain barrier options (see Example 6.12).

Lemma 5.10 (Joint distribution of Brownian motion and its maximum) Let \( M_t := \max_{s \in [0,t]} W_s \), the maximum level reached by Brownian motion in the time interval \([0, t]\). Then for \( a > 0 \), \( a \geq x \) and all \( t \geq 0 \),

\[
P\{\{M_t \geq a\} \cap \{W_t \leq x\}\} = 1 - \Phi\left(\frac{2a - x}{\sqrt{t}}\right),
\]
where \( \Phi(x) \) is the c.d.f. of standard normal distribution \( N(0, 1) \).

**Proof.** Notice that \( M_t \geq 0 \) and is non-decreasing in \( t \) and if, for \( a > 0 \), \( T_a \) is defined to be the first hitting time of level \( a \), then \( \{M_t \geq a\} = \{T_a \leq t\} \). On \( \{T_a \leq t\} \), we can take \( T = T_a \) in Lemma 5.9 (reflection principle) to obtain \( \{W_t \leq x\} = \{2W_{T_a} - \tilde{W}_t \leq x\} = \{2a - x \leq \tilde{W}_t\} \). Thus we have

\[
P(\{M_t \geq a\} \cap \{W_t \leq x\}) = P(\{T_a \leq t\} \cap \{2a - x \leq \tilde{W}_t\})
\]

\[
= P\left(\{T_a \leq t\} \cap \left\{2a - x \leq \tilde{W}_t\right\}\right)
\]

\[
= P\left(2a - x \leq \tilde{W}_t\right) \quad \text{(see explanations below)}
\]

\[
= 1 - \Phi\left(\sqrt{\frac{2a - x}{t}}\right).
\]

Note that the Reflection Principle is cleverly used in the third equality to facilitate the derivation of the joint density: As we focus on \( a \geq x \), we have \( 2a - x \geq a \). If \( 2a - x \leq \tilde{W}_t \), then necessarily \( \{W_t\} \) has hit \( a \) before time \( t \). (if \( \{W_t\} \) hasn’t hit \( a \), then from definition, \( \tilde{W}_t < a \leq 2a - x \)). Now, since \( \{W_t\} \) has hit level \( a \), \( \{T_a \leq t\} \) must have happened, so we only need to consider the probability of the event \( \{2a - x \leq \tilde{W}_t\} \). \( \square \)

With the distributional result on the first hitting time \( T_a \) in Lemma 5.8, we have the following result.
Theorem 5.11. (Divergence and Return) Brownian motion will almost surely eventually hit any and every real value (no matter how large or negative), i.e.,

\[ P \left( \sup_{s \geq 0} W_s = \infty \right) = P \left( \inf_{s \geq 0} W_s = -\infty \right) = 1. \]

Also, no matter how far above the axis, the B.M. will almost surely be back down to zero at some later time.

Proof. Note the equivalence of the events \( \{ T_a \leq t \} = \{ \sup_{s \leq t} W_s \geq a \} \). Since \( \{ \sup_{s \leq t} W_s \geq a \} \) is a contracting sequence of events as \( a \) increases, and an increasing sequence of events as \( t \) increases, we have

\[
P \left( \sup_{s \geq 0} W_s = \infty \right) = \lim_{a \to \infty} P \left( \sup_{s \geq 0} W_s \geq a \right)
= \lim_{a \to \infty} \lim_{t \to \infty} P \left( \sup_{0 \leq s \leq t} W_s \geq a \right)
= \lim_{a \to \infty} \lim_{t \to \infty} 2P \left( W_t \geq a \right) \quad \text{(Lemma 5.8)}
= \lim_{a \to \infty} 2 \left( 1 - \Phi \left( \frac{a}{\sqrt{t}} \right) \right)
= \lim_{a \to \infty} 2 \left( 1 - \Phi (0) \right)
= 1.
\]

The equality \( P \left( \inf_{s \geq 0} W_s = -\infty \right) = 1 \) can be shown similarly (Exercise 5.5).

Lastly we show the second statement. Given any \( s \), from Example 5.2, \( \tilde{W}_t := W_t + s - W_s \) is a B.M. As we just showed \( W_t \) hits any value, say \( a \), no matter large or negative, then \( \tilde{W}_t := W_t + T_a - W_{T_a} \) is a Brownian motion. The fact that \( \tilde{W}_t \) hits any value implies that \( W_t \) must return to 0. \( \square \)

5.4.3 Hitting a sloping line

For pricing a perpetual American put option, we shall use the following result.

Proposition 5.12 (Hitting an Upward Sloping Line) Set \( T_{a,b} := \inf \{ t \geq 0 : W_t = a + bt \} \), where \( T_{a,b} \) is taken to be infinity if no such time exists. Then for \( \theta > 0 \), \( a > 0 \) and \( b \geq 0 \)

\[
\mathbb{E} \exp \left\{ -\theta T_{a,b} \right\} = \exp \left\{ -a \left( b + \sqrt{b^2 + 2\theta} \right) \right\}.
\]
Before we prove Proposition 5.12, we consider the simple case \( b = 0 \).

**Lemma 5.13** Let \( W_t \) be a Brownian motion and let \( T_a := \inf\{ t \geq 0 : W_t = a \} \). Then for \( \theta > 0 \), \( a > 0 \)

\[
\mathbb{E} \exp\{-\theta T_a\} = \exp\left\{-a\sqrt{2\theta}\right\}.
\]

**Proof.** First assume that \( a > 0 \) (The case \( a < 0 \) follows by symmetry). We apply the optional stopping theorem to the martingale

\[
M_t = e^{\sigma W_t - \frac{1}{2} \sigma^2 t}.
\]

and the stopping time \( T_a \). Since \( M_{T_a \wedge n} \) is a stopped martingale, we have

\[
1 = \mathbb{E}(M_0) = \mathbb{E}(M_{T_a \wedge 0}) = \mathbb{E}(M_{T_a \wedge n}). \tag{5.6}
\]

Note that from Theorem 5.11 that \( \mathbb{P}(T_a < \infty) = 1 \). Thus, \( \lim_{n \to \infty} M_{T_a \wedge n} = M_{T_a} \). Moreover, \( M_{T_a \wedge n} = e^{\sigma W_{T_a \wedge n} - \frac{1}{2} \sigma^2 T_a \wedge n} \leq e^{\sigma a} \), so the Dominant Convergence Theorem (DCT) can be applied. To be specific, taking limit on both sides of (5.6),

\[
1 = \lim_{n \to \infty} \mathbb{E}(M_{T_a \wedge n}) \overset{D.C.T.}{=} \mathbb{E}(\lim_{n \to \infty} M_{T_a \wedge n}) = \mathbb{E}(M_{T_a}) = e^{\sigma a} \mathbb{E}(e^{-\frac{1}{2} \sigma^2 T_a}). \tag{5.7}
\]

The proof is completed by taking \( \theta = \frac{1}{2} \sigma^2 \). \( \square \)

**Proof. (Proof of Proposition 5.12)** Fix \( \theta > 0 \), and for \( a > 0 \), \( b \geq 0 \), set

\[
\psi(a, b) := \mathbb{E} \exp\{-\theta T_{a,b}\}. \tag{5.8}
\]

Note that for any \( a_1 \) and \( a_2 \) such that \( a = a_1 + a_2 \), the first hitting time for \( a + bt \) can be decomposed into the sum of two independent first hitting times of \( a_1 + bt \) and \( a_2 + bt \), i.e.,

\[
T_{a_1+a_2,b} = T_{a_1,b} + \left( T_{a_1+a_2,b} - T_{a_1,b} \right) \overset{i.i.d.}{=} T_{a_1,b} + T_{a_2,b},
\]

where \( T_{a_2,b} \) is independent of \( T_{a_1,b} \) and has the same distribution as \( T_{a_2,b} \). (see Figure 5.3). In other words,

\[
\psi(a_1 + a_2, b) = \psi(a_1, b) \psi(a_2, b),
\]

and this implies (see Exercise 5.24) that

\[
\psi(a, b) = \exp\{-k(b)a\}
\]

for some function \( k(b) \). Since \( b \geq 0 \), the process must hit the level \( a \) before it can hit the line \( x = a + bt \). Thus \( T_{a,b} \) can be decomposed into two parts: \( T_{a,b} = T_a + \tilde{T}_{bT_a,b} \), where \( T_a = \min\{ t : W_t = a \} \) (see Figure 5.4). Writing \( f_{T_a} \) for the p.d.f. of \( T_a \). Conditioning on \( T_a \), we have

\[
\]
\[
\psi(a, b) = \mathbb{E} \left( \mathbb{E} \left( \exp \left\{ -\theta T_{a,b} \right\} \mid T_a \right) \right)
\]
\[
= \int_0^\infty f_T(t) \mathbb{E} \left( \exp \left\{ -\theta t \right\} \mid T_a = t \right) dt
\]
\[
= \int_0^\infty f_T(t) \exp \left\{ -\theta t \right\} \mathbb{E} \left( \exp \left\{ -\theta T_{b,t} \right\} \right) dt
\]
\[
= \int_0^\infty f_T(t) \exp \left\{ -\theta t \right\} \exp \left\{ -k(b)t \right\} dt
\]
\[
= \mathbb{E} \exp \left\{ -[\theta + k(b)b] T_a \right\}
\]
\[
= \exp \left\{ -a \sqrt{2(\theta + k(b)b)} \right\}.
\]

We now have two expressions for \( \psi(a, b) \). Equating them gives
\[
k^2(b) = 2\theta + 2k(b)b,
\]
Note from (5.8) that for \( \theta > 0 \) we must have \( \psi(a, b) \leq 1 \). Thus \( k(b) \) must be positive, and we choose
\[
k(b) = b + \sqrt{b^2 + 2\theta},
\]
which completes the proof. \(\square\)

Fig. 5.3 In the notation of Proposition 5.12, \( T_{a_1+a_2,b} = T_{a_1,b} + \tilde{T}_{a_2,b} \) where \( \tilde{T}_{a_2,b} \) has the same distribution as \( T_{a_2,b} \).
5.5 Variation of Brownian Motions

Fig. 5.4 In the notation of Proposition 5.12, $T_{a,b} = T_a + \tilde{T}_{b,T_a}$ where $\tilde{T}_{b,T_a}$ has the same distribution as $T_{b,T_a}$.

Remark 5.4. For a real constant $\mu$, we refer to the process $W_t^\mu := W_t + \mu t$ as a Brownian motion with drift $\mu$. Then, in the notation above, $T_{a,b}$ can be interpreted as the first hitting time of the level $a$ by a Brownian motion with drift $-b$.

5.5 Variation of Brownian Motions

The notion of variation of a process will be useful in defining stochastic integrals in the next chapter.

Definition 5.10. (Variation of a function) Let $\Pi = (0 = t_0, t_1, \ldots, t_{N(\Pi)} = T)$ be a partition of the interval $[0, T]$, $N(\Pi)$ be the number of intervals partitioned by $\Pi$, and $\delta(\Pi)$ be the mesh of $\Pi$, i.e., the length of the longest interval in $\Pi$,

$$\delta(\Pi) = \max_{i=1,\ldots,N(\Pi)} |t_i - t_{i-1}|.$$  \hspace{1cm} (5.9)

Then, the $p$-variation of a function $f$ is defined as

$$\lim_{\delta \to 0} \left\{ \sup_{\Pi, \delta(\Pi) = \delta} \sum_{i=1}^{N(\Pi)} |f(t_i) - f(t_{i-1})|^p \right\}.$$  \hspace{1cm} (5.9)
First we show the convergence of the second variation, or quadratic variation, of Brownian motion.

**Theorem 5.14. (Quadratic Variation of Brownian Motion)** Let \( W_t \) be a Brownian motion. Define the quadratic variation (2-variation) w.r.t. \( \Pi \) by

\[
S(\Pi) = \sum_{j=1}^{N(\Pi)} \left| W_{t_j} - W_{t_{j-1}} \right|^2,
\]

(5.10)

If \( \Pi_n \) is a sequence of partitions of \([0, T]\) such that \( \delta(\Pi_n) \to 0 \), then

\[
E\left| S(\Pi_n) - T \right|^2 \to 0.
\]

(5.11)

**Proof.** To stress on the dependence of \( t_j \)s on \( \Pi_n \), we use \( t_{n,j} \) to denote the \( t_j \) of \( \Pi_n \).

Since \( T = \sum_{j=1}^{N(\Pi_n)} t_{n,j} - t_{n,j-1} \), we have

\[
\left| S(\Pi_n) - T \right|^2 = \left| \sum_{j=1}^{N(\Pi_n)} \left( W_{t_{n,j}} - W_{t_{n,j-1}} \right)^2 - (t_{n,j} - t_{n,j-1}) \right|^2
\]

(5.12)

where

\[
\delta_{n,j} = \left| W_{t_{n,j}} - W_{t_{n,j-1}} \right|^2 - (t_{n,j} - t_{n,j-1}).
\]

Since Brownian motion has independent increment, we have

\[
E(\delta_{n,j} \delta_{n,k}) = E(\delta_{n,j})E(\delta_{n,k}) = 0, \quad \text{if} \quad j \neq k.
\]

(5.13)

Also, using the fact that \( E(X^2) = \sigma^2 \) and \( E(X^4) = 3\sigma^4 \) for \( X \sim N(0, \sigma^2) \) (Exercise 5.11), we have

\[
E(\delta_{n,j}^2) = E \left| W_{t_{n,j}} - W_{t_{n,j-1}} \right|^4 - 2|W_{t_{n,j}} - W_{t_{n,j-1}}|^2(t_{n,j} - t_{n,j-1}) + (t_{n,j} - t_{n,j-1})^2
\]

\[
= 3(t_{n,j} - t_{n,j-1})^2 - 2(t_{n,j} - t_{n,j-1})^2 + (t_{n,j} - t_{n,j-1})^2
\]

= 2(t_{n,j} - t_{n,j-1})^2.
\]

(5.14)

Combining (5.13) and (5.14), taking expectation on both sides of (5.12) gives
5.5 Variation of Brownian Motions

\[
E[\mathcal{S}(\Pi_n) - T]^2 = \sum_{j=1}^{N(\Pi_n)} E(\delta_{n,j}^2) + 2 \sum_{j<k} E(\delta_{n,j} \delta_{n,k}) \\
= \sum_{j=1}^{N(\Pi_n)} 2(t_{n,j} - t_{n,j-1})^2 \\
\leq 2\delta(\Pi_n) \sum_{j=1}^{N(\Pi_n)} (t_{n,j} - t_{n,j-1}) \\
= 2\delta(\Pi_n)T \to 0,
\]

since \( T \) is fixed and \( \delta(\Pi_n) \to 0 \). Thus the proof is completed. \( \Box \)

Motivated by the above theorem, we have the following general definition.

**Definition 5.11.** (Quadratic Variation Process) Suppose that \( \{M_t\} \) is a martingale. The quadratic variation process associated with \( \{M_t\}_{t \geq 0} \) is the process \( \{[M]_t\}_{t \geq 0} \) such that for any sequence of partition \( \{\Pi_n\} \) of \([0, T]\) with \( \delta(\Pi_n) \to 0 \),

\[
E \left[ \sum_{j=1}^{N(\Pi_n)} |M_{t_j} - M_{t_{j-1}}|^2 - [M]_T \right] \to 0,
\]

as \( n \to \infty \).

**Corollary 5.15** Theorem 5.14 shows that the quadratic variation process of standard Brownian motion \( W_t \) is

\[ [W]_T = T. \]

**Corollary 5.16** (First Variation of Brownian Motion) The first variation of Brownian motion is \( \infty \) for any interval \([0, T]\).

**Proof.** Suppose on the contrary that the first variation is not \( \infty \), then there exist some \( T \) and some \( K \) such that

\[
\lim_{\delta \to 0} \left\{ \sup_{\Pi : \delta(\Pi) = \delta} \sum_{j=1}^{N(\Pi)} |W_{t_j} - W_{t_{j-1}}| \right\} = K < \infty,
\]

where \( \Pi \) is a partition of \([0, T]\). Then

\[
\lim_{\delta \to 0} \left\{ \sup_{\Pi : \delta(\Pi) = \delta} \sum_{j=1}^{N(\Pi)} |W_{t_j} - W_{t_{j-1}}|^2 \right\} \leq \lim_{\delta \to 0} \left\{ \sup_{\Pi : \delta(\Pi) = \delta} \sum_{j=1}^{N(\Pi)} |W_{t_j} - W_{t_{j-1}}| \right\} \sup_{\Pi : \delta(\Pi) = \delta} \sum_{j=1}^{N(\Pi)} |W_{t_j} - W_{t_{j-1}}| \\
\leq K \lim_{\delta \to 0} \left\{ \sup_{\Pi : \delta(\Pi) = \delta} |W_{t_j} - W_{t_{j-1}}| \right\} \\
\to 0 \quad \text{(continuity of } W_t\text{)},
\]
contradicting Corollary 5.15 that the quadratic variation of B.M. is $T$. Thus the first variation of B.M. is $\infty$.

5.6 Exercises

Exercise 5.1. Show that the process $\tilde{W}_t$ in Example 5.2 is a standard Brownian motion.

Exercise 5.2. Based on the probability measure in Definition 5.1, can you deduce property 1 and 2 in Definition 5.2 and the transition probability in Definition 5.3?

Exercise 5.3. Using integration by parts, prove that if $\{W_t : t \geq 0\}$ is a standard Brownian motion under $P$, for $x > 0$,

$$P\{W_t \geq x\} \equiv \int_x^\infty \frac{1}{\sqrt{2\pi t}} \exp \left\{ \frac{-y^2}{2t} \right\} \, dy \leq \frac{\sqrt{T}}{x\sqrt{2\pi}} \exp \left\{ -\frac{x^2}{2T} \right\}.$$

Exercise 5.4. Find the third variation of a standard Brownian motion.

Exercise 5.5. If $W_t$ is a Brownian motion, show that $P(\inf_{s \geq 0} W_s = -\infty) = 1$.

Exercise 5.6. Prove Lemma 5.1.

Exercise 5.7. Let $Z$ be normally distributed with mean zero and variance one under the measure $P$. What is the distribution of $\sqrt{T}Z$? Is the process $X_t := \sqrt{T}Z$ a Brownian motion?

Exercise 5.8. Suppose that $W_t$ and $\tilde{W}_t$ are independent Brownian motions under the measure $P$ and let $\rho \in [-1, 1]$ be a constant. Is the process $X_t := \rho W_t + \sqrt{1 - \rho^2} \tilde{W}_t$ a Brownian motion?

Exercise 5.9. Let $\{W_t : t \geq 0\}$ be standard Brownian motion under the measure $P$. Which of the following are $P$-Brownian motions?

1. $\{-W_t : t \geq 0\}$,
2. $\{cW_t^2 : t \geq 0\}$, where $c$ is some non-zero constant,
3. $\{\sqrt{T}W_t : t \geq 0\}$,
4. $\{W_{2t} - W_t : t \geq 0\}$.

Justify your answers.

Exercise 5.10. Let $\{W_t : t \geq 0\}$ be standard Brownian motion under the measure $P$, and let $\{\mathcal{F}_t\}, t \geq 0$, be filtration for this Brownian motion. Show that $W_{t^2} - t$ is a $\{P, \{\mathcal{F}_t\}\}$ martingale.

Exercise 5.11. Suppose that $X$ is normally distributed with mean $\mu$ and variance $\sigma^2$. Calculate $E\exp\{\theta X\}$ and hence evaluate $EX^2$ and $EX^4$. 
Exercise 5.12. Brownian motion is not adequate as a stock market model. First, it has constant mean, whereas the company stocks usually grow at some rates. Moreover, it may be too ‘noisy’ (variance of the path increments is large) or not noisy enough. We can scale to change the ‘noisiness’ and we can artificially introduce a drift, but this is still not a good model, since it may take negative values. Suppose that \( \{W_t : t \geq 0\} \) is standard Brownian motion under \( \mathbb{P} \). Define a new process \( \{S_t : t \geq 0\} \) by \( S_t := \mu t + \sigma W_t \) where \( \sigma > 0 \) and \( \mu \in \mathbb{R} \) are constants. Show that for all values of \( \sigma > 0 \), \( \mu \in \mathbb{R} \) and \( T > 0 \) there is a positive probability that \( S_T \) is negative.

Exercise 5.13. Let \( \{W_t : t \geq 0\} \) be a standard Brownian motion and \( \{\mathcal{F}_t^W\} \) be the natural filtration of the B.M.. Show that \( e^{\sigma W_t - \frac{\sigma^2}{2} t} \) is a \( (\mathbb{P}, \{\mathcal{F}_t^W\}) \) martingale.

Exercise 5.14. Let \( \{W_t : t \geq 0\} \) be standard Brownian motion under \( \mathbb{P} \). For \( a > 0 \), let \( T_a \) be the ‘hitting time of level \( a \)’, i.e., \( T_a = \inf \{t \geq 0 : W_t = a\} \). Use the fact \( \mathbb{E}\exp \{-\theta T_a\} = \exp \{-a \sqrt{2\theta} \} \) to calculate
1. \( \mathbb{E}T_a \),
2. \( \mathbb{P}\{T_a < \infty\} \).

Exercise 5.15. Let \( \{W_t : t \geq 0\} \) be standard Brownian motion under \( \mathbb{P} \) and define \( \{M_t : t \geq 0\} \) by \( M_t := \max_{s \in [0,t]} W_s \). Suppose that \( x \geq a \). Calculate
1. \( \mathbb{P}\{M_t \geq a \} \cap \{W_t \geq x\} \),
2. \( \mathbb{P}\{M_t \geq a \} \cap \{W_t \leq x\} \).

Exercise 5.16. Let \( \{W_t : t \geq 0\} \) be standard Brownian motion under \( \mathbb{P} \) and define \( \{M_t : t \geq 0\} \) by \( M_t := \max_{s \in [0,t]} W_s \).
1. Show that \( M_t, |W_t| \) and \( M_t - W_t \) have the same marginal distribution, with density
   \[
   f_{M_t}(z) = \frac{2}{\sqrt{t}} \phi \left( \frac{z}{\sqrt{t}} \right) 1_{(0,\infty)}(z),
   \]
   where \( \phi(\cdot) \) is the p.d.f. of \( N(0,1) \).
2. \( \mathbb{E}M_t = \sqrt{2} \sqrt{t} \).

Exercise 5.17. Find the probability density function of the stopping time \( T_a = \inf \{t \geq 0 : W_t = a\} \).

Exercise 5.18. Let \( \{W_t : t \geq 0\} \) be standard Brownian motion under \( \mathbb{P} \). Derive the joint distribution of \( W_t \) and \( m_t := \min_{s \in [0,t]} W_s \). (Hint: Consider \( -W_t \).)

Exercise 5.19. Let \( \{W_t : t \geq 0\} \) be standard Brownian motion under \( \mathbb{P} \), and \( s \in [0,t) \). Show that the conditional distribution of \( W_s \) given \( W_t = b \) is Normal and give its mean and variance.
Exercise 5.20. Let $(X, Y)$ be a pair of random variables with joint density function

$$f_{X,Y}(x, y) = \frac{2|x| + y}{\sqrt{2\pi}} \exp \left\{ -\frac{(2|x| + y)^2}{2} \right\} 1_{\{y \geq -|x|\}}.$$

Show that $X$ and $Y$ are standard normal random variables and that they are uncorrelated but not independent.

Exercise 5.21. Let $\{W_t : t \geq 0\}$ be standard Brownian motion under $\mathbb{P}$. For the Gaussian process

$$Y_t = \exp \{-\alpha t\} W_{\exp\{2\alpha t\}} ,$$

find its mean and covariance functions, $\mu(t) = \mathbb{E}(Y_t)$ and $\gamma_k = \text{Cov}(Y_t, Y_{t+k})$.

Exercise 5.22. In attempt to show the almost surely continuity of Brownian motion, Keith provided the following argument:

$$\mathbb{P}(\lim_{t \to 0} W_t = 0) = \lim_{n \to \infty} \mathbb{P} \left( \cap_{n \geq 1} \left\{ \lim_{t \to 0} |W_t| < \frac{1}{n} \right\} \right)$$

$$= \lim_{n \to \infty} \mathbb{P} \left( \left\{ \lim_{t \to 0} |W_t| < \frac{1}{n} \right\} \right)$$

$$= \lim_{n \to \infty} \mathbb{P} \left( \left\{ \text{some } t^*, \forall t \leq t^*, |W_t| < \frac{1}{n} \right\} \right)$$

$$= \lim_{n \to \infty} \mathbb{P} \left( \bigcup_{t^* \in \mathbb{R}} \cap_{t \leq t^*} \left\{ |W_t| < \frac{1}{n} \right\} \right)$$

$$\geq \lim_{n \to \infty} \mathbb{P} \left( \bigcup_{m=1}^{\infty} \cap_{t \leq 1/m} \left\{ |W_t| < \frac{1}{n} \right\} \right)$$

$$= \lim_{n \to \infty} \lim_{m \to \infty} \mathbb{P} \left( \left\{ |W_{1/m}| < \frac{1}{n} \right\} \right)$$

$$\geq \lim_{n \to \infty} \lim_{m \to \infty} \left[ \Phi(\sqrt{m}/n) - \Phi(-\sqrt{m}/n) \right]$$

$$\geq 1.$$

Is the argument correct? Explain.

Exercise 5.23. In this exercise we describe a second proof of the fact $\mathbb{P}(\sup_{t \geq 0} W_t = \infty) = 1$. Let $Z := \sup_{t \in [0, \infty)} W_t$.

1. Show that for any $c > 0$, $cZ$ has the same distribution as $Z$.
2. Using the above result, show that $P(Z \in [1, n)) = 0$ and $P(Z \in (1/n, 1]) = 0$ for all $n \geq 1$. Thus show that with probability one, $Z \in \{0, \infty\}$.
3. By noting that \( \{ Z = 0 \} \subset \{ W_1 < 0 \} \cap \{ \sup_{t \geq 0} W_{t+1} - W_1 = 0 \} \) and that \( W_{t+1} - W_1 \) is a B.M., argue that

\[
P(Z = 0) \leq P(W_1 < 0)P(Z = 0) = P(W_1 < 0)P(Z = 0).
\]

Hence deduce that \( P(Z = \infty) = 1 \).

4. Using similar argument, show that \( P(\inf_{t \in [0, \infty)} W_t = -\infty) = 1 \).

**Exercise 5.24.** Show that if a continuous function \( f(x) \) satisfies \( f(a+b) = f(a)f(b) \)
and \( f(1) \neq 0 \), then \( f(x) = e^{cx} \) for some \( c \).