Chapter 4
Martingales in Discrete Time

4.1 Sequences of Random Variables

A sequence of random variables $X_1, X_2, \ldots$ that represents the outcomes of a series of random phenomena is called a discrete time stochastic process. Examples include a sequence of coin tosses or the evolution of daily stock closing prices. Although the random variables in a discrete time stochastic process are indexed by natural numbers, these natural numbers are not necessarily related to the physical time of event occurrences. In other words, the ‘discrete time’, $t = 1, 2, 3, \ldots$ is used to keep track of the order of events, and may not be evenly spaced in physical time. For example, the indexes for stock prices are recorded only on business days, but not on Saturdays, Sundays or public holidays.

Definition 4.1. (Discrete Time Stochastic Process) A discrete time Stochastic Process is an infinite dimensional random vector $X = \{X_t\}_{t=1,2,\ldots}$ from some abstract measurable space $(\Omega, \mathcal{F})$ to the measurable space $(\mathbb{R}^\infty, \mathcal{B}^\infty)$. Here $\mathcal{B}^\infty$ is understood as $\sigma(\mathcal{B} \times \mathcal{B} \times \ldots)$.

Definition 4.2. (Sample Path) For any fixed $\omega \in \Omega$, the image of the stochastic process is a sequence of numbers $X_1(\omega), X_2(\omega), \ldots$, which is called a sample path.

Example 4.1. Consider three consecutive coin tosses $X_i$, $i = 1, 2, 3$, with $X_i = 1$ for head and 0 for tail. Setting $S = \{0, 1\}$, we can take $\Omega = S \times S \times S$, $\mathcal{F} = \sigma(2^S \times 2^S \times 2^S)$. Note that $\Omega$ contains 8 possible elements, and each element corresponds to a sample path. For example, if $\omega_1 = HHH$, $\omega_2 = HHT$, $\omega_8 = TTT$, then $X_1(\omega_1) = X_2(\omega_1) = X_3(\omega_1) = 1$; $X_1(\omega_2) = X_2(\omega_2) = 1, X_3(\omega_2) = 0$; $X_1(\omega_8) = X_2(\omega_8) = X_3(\omega_8) = 0$. A probability measure may be defined by $P(\{\omega_i\}) = \frac{1}{8}$ for $i = 1, 2, \ldots, 8$. 
4.1.1 Filtrations

Intuitively, the $\sigma$-field $\mathcal{B}_\infty$ in Definition 4.1 contains all information in the whole process $X_1, X_2, \ldots$. If the time $n$ goes on sequentially, we can define the filtration, which is an increasing sequence of $\sigma$-fields that contains the knowledge acquired up to a time point.

**Definition 4.3. (Filtration)** A sequence of $\sigma$-fields $\mathcal{F}_1, \mathcal{F}_2, \ldots$ on $\Omega$ such that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ is called a filtration.

**Definition 4.4. (Random Variables adapted to a Filtration)** A sequence of random variables $X_1, X_2, \ldots$ are adapted to a filtration $\mathcal{F}_1, \mathcal{F}_2, \ldots$ if $X_n$ is $\mathcal{F}_n$-measurable for each $n = 1, 2, \ldots$.

**Example 4.2.** If $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ is the $\sigma$-field generated by $X_1, \ldots, X_n$, then $X_1, X_2, \ldots$ is adapted to $\mathcal{F}_1, \mathcal{F}_2, \ldots$. Such filtration $\{\mathcal{F}_n\}_{n=1,\ldots}$ is known as the natural filtration.

**Remark 4.1. ($\sigma$-field = Information)** The $\sigma$-field $\mathcal{F}_n$ represents our knowledge up to time $n$ about everything related to $X_1, \ldots, X_n$. (In general, it may and often does contain more information). To understand it mathematically, note that if $A \in \mathcal{F}_n$, then $\mathbb{P}(A|\mathcal{F}_n) = E(1_A|\mathcal{F}_n) = 1_A$, since $1_A$ is $\mathcal{F}_n$ measurable. That is, at time $n$, if $A \in \mathcal{F}_n$, then we know that whether $A$ has occurred or not, i.e.,

$$\mathbb{P}(A|\mathcal{F}_n)(\omega) = 1_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

(A occurred)

(A did not occur).

**Example 4.3.** For a sequence $X_1, X_2, \ldots$ of coin tosses we take $\mathcal{F}_n$ to be the $\sigma$-field generated by $X_1, \ldots, X_n$, i.e.,

$$\mathcal{F}_n = \sigma(X_1, \ldots, X_n).$$

Let

$$A = \{\text{the first 5 tosses produce at least 2 heads}\}.$$  

At time $n = 5$, the coin has been tossed five times, it will be possible to decide whether $A$ has occurred or not. This means that $A \in \mathcal{F}_5$. However, at $n = 4$ it is not always possible to tell if $A$ has occurred or not. If the outcomes of the first four tosses are, say,

$$\text{tail, tail, head, tail}, \quad (4.1)$$

then the event $A$ remains undecided. Therefore, $A \notin \mathcal{F}_4$. $\square$

**Remark 4.2.** Suppose that the outcomes of the first four coin tosses are
tail, head, tail, head.

In this case it is possible to tell that $A$ has occurred already at $n = 4$, regardless of the outcome of the fifth toss. It does not mean, however, that $A$ belongs to $\mathcal{F}_4$. Intuitively, in order for $A$ to belong to $\mathcal{F}_4$, it must be possible to tell whether $A$ has occurred or not after the first four tosses, no matter what the first four outcomes are. This is clearly not so in view of (4.1).

Mathematically, $A = \{HHTT, HTHT, \ldots\}$ ($C_5^2 + C_5^3 + C_5^4 + C_5^5$ elements) is clearly not in $\mathcal{F}_4$. On the other hand, the statement “$A$ must occur given that the first four coin tosses were $\{THTH\}$” is equivalent to the statement “$\mathbb{P}(A|\{THTH\}) = 1$”, which is not the same as the statement “$A$ belongs to $\mathcal{F}_4$”, i.e., “$\mathbb{P}(A|\mathcal{F}_4)(\omega) = 1_A(\omega)$”. Note that the last statement is not correct.

\section*{4.2 Martingales}

The concept of a martingale has its origin in gambling, namely, it describes a fair game of chance. Similarly, the notions of submartingale and supermartingale defined below are related to favorable and unfavorable games of chance. In fact, martingales reach well beyond gambling and appear in various areas of modern probability and stochastic analysis. First we introduce the basic definitions and properties for martingales in discrete time.

\begin{definition}[Martingale]
A sequence $X_1, X_2, \ldots$ of random variables is called a martingale with respect to a filtration $\mathcal{F}_1, \mathcal{F}_2, \ldots$ if
\begin{enumerate}
\item $X_n$ is integrable for each $n = 1, 2, \ldots$, i.e., $X_n \in L^1$, or $E|X_n| < \infty$.
\item $X_1, X_2, \ldots$ is adapted to $\mathcal{F}_1, \mathcal{F}_2, \ldots$;
\item $E(X_{n+1} | \mathcal{F}_n) = X_n$ a.s. for each $n = 1, 2, \ldots$.
\end{enumerate}
\end{definition}

\begin{example}
Let $Z_1, Z_2, \ldots$ be a sequence of independent integrable random variables such that $E(Z_n) = 0$ for all $n = 1, 2, \ldots$. Put
\begin{align*}
X_n &= Z_1 + \cdots + Z_n, \\
\mathcal{F}_n &= \sigma(Z_1, \ldots, Z_n).
\end{align*}
Then $X_n$ is adapted to the filtration $\mathcal{F}_n$, and is integrable because
\begin{align*}
E(|X_n|) &= E(|Z_1 + \cdots + Z_n|) \\
&\leq E(|Z_1|) + \cdots + E(|Z_n|) \\
&< \infty,
\end{align*}
since each $Z_i$ is integrable. Moreover,
\[ E(X_{n+1} | \mathcal{F}_n) = E(Z_{n+1} | \mathcal{F}_n) + E(X_n | \mathcal{F}_n) \]
\[ = E(Z_{n+1}) + X_n \]
\[ = X_n, \]

since \( Z_{n+1} \) is independent of \( \mathcal{F}_n \) (condition on independent \( \sigma \)-field reduces to unconditional) and \( X_n \) is \( \mathcal{F}_n \)-measurable (taking out what is known); see Theorem 3.6. Therefore, \( X_n \) is a martingale with respect to \( \mathcal{F}_n \). □

**Example 4.5.** Let \( X \) be an integrable random variable and let \( \mathcal{F}_1, \mathcal{F}_2, \ldots \) be a filtration. For \( n = 1, 2, \ldots \), put
\[ X_n = E(X | \mathcal{F}_n). \]

By construction, \( X_n \) is \( \mathcal{F}_n \)-measurable. Next, since \( |X_n| = |E(X | \mathcal{F}_n)| \leq E(|X| | \mathcal{F}_n) \), we have by Tower property that
\[ E(|X_n|) \leq E(E(|X| | \mathcal{F}_n)) = E(|X|) < \infty. \]

Lastly, by \( \mathcal{F}_n \subset \mathcal{F}_{n+1} \) and the Tower property of conditional expectation,
\[ E(X_{n+1} | \mathcal{F}_n) = E(E(X | \mathcal{F}_{n+1}) | \mathcal{F}_n) = E(X | \mathcal{F}_n) = X_n. \]

Therefore \( X_n \) is a martingale with respect to \( \mathcal{F}_n \). □

**Example 4.6. (Symmetric Random Walk)** Let \( X_n \) be a symmetric random walk, that is,
\[ X_n = Z_1 + \cdots + Z_n, \]
where \( Z_1, Z_2, \ldots \) is a sequence of independent identically distributed random variables such that
\[ P\{Z_n = 1\} = P\{Z_n = -1\} = \frac{1}{2} \]
(a sequence of coin tosses, for example). Then \( X_n^2 - n \) is a martingale with respect to the filtration
\[ \mathcal{F}_n = \sigma(Z_1, \ldots, Z_n). \]

It is standard to check the measurability and integrability of \( X_n \) similar to Example 4.4 (See Exercise 4.13). Therefore, it remains to verify that \( E(X_{n+1}^2 - (n+1) | \mathcal{F}_n) = X_n^2 - n \). Write
\[ X_{n+1}^2 = (X_n + Z_{n+1})^2 = Z_{n+1}^2 + 2Z_{n+1}X_n + X_n^2. \]

Since \( X_n \) is \( \mathcal{F}_n \)-measurable and \( Z_{n+1} \) is independent of \( \mathcal{F}_n \), we have
\[ E(X_{n+1}^2 | \mathcal{F}_n) - (n+1) = E(Z_{n+1}^2 | \mathcal{F}_n) + 2E(Z_{n+1}X_n | \mathcal{F}_n) + E(X_n^2 | \mathcal{F}_n) - (n+1) \]
\[ = E(Z_{n+1}^2) + 2X_nE(Z_{n+1}) + X_n^2 - (n+1) \]
\[ = X_n^2 - n, \]
as desired. □
4.3 Trading Strategies and Martingales

**Definition 4.6.** We say that $X_1, X_2, \ldots$ is a supermartingale (submartingale) with respect to a filtration $\mathcal{F}_1, \mathcal{F}_2, \ldots$ if

1. $X_n$ is integrable for each $n = 1, 2, \ldots$;
2. $X_1, X_2, \ldots$ is adapted to $\mathcal{F}_1, \mathcal{F}_2, \ldots$;
3. $E(X_{n+1}|\mathcal{F}_n) \leq X_n$ (respectively, $E(X_{n+1}|\mathcal{F}_n) \geq X_n$) a.s. for each $n = 1, 2, \ldots$.

**Example 4.7.** Let $X_n$ be a sequence of square integrable random variables. If $X_n$ is a martingale with respect to a filtration $\mathcal{F}_n$, then $X_n^2$ is a submartingale with respect to the same filtration. To see this, note that

1. $E[X_n^2] < \infty$ by the square integrability assumption.
2. $X_n^2$ is $\mathcal{F}_n$ measurable since $X_n$ is. So $\{X_n^2\}$ is adapted to $\{\mathcal{F}_n\}$.
3. Using the Jensen’s inequality with convex function $\varphi(x) = x^2$ yields
   $$E(X_{n+1}^2|\mathcal{F}_n) \geq [E(X_{n+1}|\mathcal{F}_n)]^2 = X_n^2.$$ Thus the definition of a submartingale is verified.

4.3 Trading Strategies and Martingales

Suppose that you take part in a game with $Z_n$ being the profits (or losses) per unit of investment in period $n$. Let $Z_1, Z_2, \ldots$ be a sequence of integrable random variables. If we keep our investment to be $1 in each period, then your total profit $X_n$ after $n$ periods is

$$X_n = Z_1 + \cdots + Z_n. \quad (4.2)$$

Consider the filtration $\{\mathcal{F}_n\}_{n=0,1,\ldots}$ defined by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and

$$\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n).$$

If we are at the end of the $(n-1)$-th period, our accumulated knowledge will be represented by the $\sigma$-field $\mathcal{F}_{n-1}$. Assuming that the risk free rate is 0, the market is fair if $X_n$ is a martingale w.r.t. $\{\mathcal{F}_n\}$. In particular, for $X_0 = 0$ and $n \geq 1$, we have

$$E(X_n|\mathcal{F}_{n-1}) = X_{n-1}.$$ Thus, you expect that your total profit at time $n$ is on average the same as that at time $n - 1$. The market will be favourable to you if $X_n$ is a sub-martingale w.r.t. $\{\mathcal{F}_n\}$,

$$E(X_n|\mathcal{F}_{n-1}) \geq X_{n-1},$$

and unfavourable to you if $X_n$ is a super-martingale w.r.t. $\{\mathcal{F}_n\}$,

$$E(X_n|\mathcal{F}_{n-1}) \leq X_{n-1},$$
for \( n = 1, 2, \ldots \).

Suppose that you keep the investment to be \( \alpha_n \) at period \( n, n = 1, 2, \ldots \) (\( \alpha_n \) may be zero if you refrain from investing or negative if do short selling.) At the beginning of the \( n \)-th period, we decide the investment \( \alpha_n \) with the knowledge about the outcomes in the first \( n - 1 \) periods. Therefore it is reasonable to assume that \( \alpha_n \) is \( \mathcal{F}_{n-1} \)-measurable, where \( \mathcal{F}_{n-1} \) represents our knowledge accumulated up to and including time \( n - 1 \). Note that \( \mathcal{F}_0 = \{ \emptyset, \Omega \} \) indicates that nothing is known before the first game.

**Definition 4.7. (Trading Strategy)** A trading strategy \( \alpha_1, \alpha_2, \ldots \) (with respect to a filtration \( \mathcal{F}_1, \mathcal{F}_2, \ldots \)) is a sequence of random variables such that \( \alpha_n \) is \( \mathcal{F}_{n-1} \)-measurable for each \( n = 1, 2, \ldots \), where \( \mathcal{F}_0 = \{ \emptyset, \Omega \} \).

**Remark 4.3. (Previsible)** A sequence \( \{ \alpha_n \}_{i=1,2,\ldots} \) is called previsible if \( \alpha_n \) is \( \mathcal{F}_{n-1} \)-measurable. In general, we require all trading strategies to be previsible.

If one follows a strategy \( \alpha_1, \alpha_2, \ldots \), then \( \alpha_k \) unit of investment is made in period \( k \). The profit in the \( k \)-th period is \( \alpha_k (X_k - X_{k-1}) \), and the total profits after \( n \) periods will be

\[
Y_n = \alpha_1 (X_1 - X_0) + \cdots + \alpha_n (X_n - X_{n-1}).
\]

We also put \( Y_0 = 0 \) for convenience.

The following proposition has important consequences for traders. It means that no matter which trading strategy is used,

- a fair market will always be a fair one if the available capital and credit limit (i.e., \( \alpha_n \)'s) are bounded.
- it is impossible to turn an unfavorable market into a favorable (or vice versa) one if \( \alpha_n \)'s are non-negative.

In other words, one cannot beat the system!

**Proposition 4.1** Let \( \alpha_1, \alpha_2, \ldots \) be a previsible trading strategy, \( Z_n \) be the profits of game \( n \), \( X_n = \sum_{i=1}^{n} Z_i \) and \( Y_n = \sum_{i=1}^{n} \alpha_i Z_i \) be the total profits defined in (4.3).

1. If \( \alpha_1, \alpha_2, \ldots \) is a bounded sequence and \( X_0, X_1, X_2, \ldots \) is a martingale, then \( Y_1, Y_2, \ldots \) is a martingale (a fair market is always fair);
2. If \( \alpha_1, \alpha_2, \ldots \) is a non-negative bounded sequence and \( X_0, X_1, X_2, \ldots \) is a supermartingale, then \( Y_0, Y_1, Y_2, \ldots \) is a supermartingale (an unfavorable market is always unfavourable to any strategy).
3. If \( \alpha_1, \alpha_2, \ldots \) is a non-negative bounded sequence and \( X_0, X_1, X_2, \ldots \) is a submartingale, then \( Y_0, Y_1, Y_2, \ldots \) is a submartingale (a favorable market is always favourable to any strategy).

**Proof.** Since \( \alpha_n \) and \( Y_{n-1} \) are \( \mathcal{F}_{n-1} \)-measurable, we can take them out of the expectation conditioned on \( \mathcal{F}_{n-1} \) (‘taking out what is known’). Thus, we obtain

\[
E(Y_n|\mathcal{F}_{n-1}) = E(Y_{n-1} + \alpha_n (X_n - X_{n-1})|\mathcal{F}_{n-1}) = Y_{n-1} + \alpha_n (E(X_n|\mathcal{F}_{n-1}) - X_{n-1}).
\]
If $X_n$ is a martingale, then
\[ \alpha_n(E(X_n|\mathcal{F}_{n-1}) - X_{n-1}) = 0, \]
which proves assertion 1. If $X_n$ is a supermartingale and $\alpha_n \geq 0$, then
\[ \alpha_n(E(X_n|\mathcal{F}_{n-1}) - X_{n-1}) \leq 0, \]
proving assertion 2. Finally, if $X_n$ is a submartingale and $\alpha_n \geq 0$, then
\[ \alpha_n(E(X_n|\mathcal{F}_{n-1}) - X_{n-1}) \geq 0 \]
and assertion 3 follows.

### 4.4 Stopping Times

In any market, one can quit at any time. Let $\tau$ be the number of period you engaged before quitting the market. It can be fixed to be $\tau = 10$ (say) if one decides in advance to stop investing after 10 periods no matter what happens. But in general the decision whether to quit or not can be made after each period, depending on the knowledge accumulated so far. Therefore $\tau$ can be assumed to be a random variable taking values from the set $\{1, 2, \ldots \} \cup \{\infty\}$. Infinity is included to cover the theoretical possibility that the market never stops. At each step $n$ one should be able to decide whether to stop trading or not, i.e., whether $\tau = n$. Therefore the event that $\tau = n$ should be in the $\sigma$-field $\mathcal{F}_n$ that represents our knowledge at time $n$. This gives rise to the following definition.

**Definition 4.8. (Stopping Time)** A random variable $\tau$ with values in the set $\{1, 2, \ldots \} \cup \{\infty\}$ is called a **stopping time** (with respect to a filtration $\{\mathcal{F}_n\}$) if for each $n = 1, 2, \ldots$
\[ \{\tau = n\} \in \mathcal{F}_n. \]

**Remark 4.4.** From Definition 4.8, we have that if $\tau$ is a stopping time, then $\{\tau \leq n\} = \bigcup_{k=1}^n \{\tau = k\} \in \mathcal{F}_n$ and $\{\tau > n\} = \{\tau \leq n\}^c \in \mathcal{F}_n$.

**Example 4.8. (First Hitting Time)** Suppose that a coin is tossed repeatedly and you win (lose) $1, if a head (tail) is landed. Suppose that you start investing with, say, $5 in your pocket and decide to play until you have $10 or you lose everything. If $X_n$ is the amount you have at step $n$, then the time when you stop the game is
\[ \tau = \inf\{n : X_n = 10 \text{ or } 0\}, \]
and is called the **first hitting time** of the random sequence $X_n$. This $\tau$ is a stopping time with respect to the filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$: To see this, note that
\[ \{\tau = n\} = \{0 < X_1 < 10\} \cap \cdots \cap \{0 < X_{n-1} < 10\} \cap \{X_n = 10 \text{ or } 0\}. \]
Now each of the sets on the right-hand side belongs to $\mathcal{F}_n$, so does their intersection too. This implies that
\[
\{ \tau = n \} \in \mathcal{F}_n
\]
for each $n$, i.e., $\tau$ is a stopping time. $\square$

Let $X_n$ be a sequence of random variables adapted to a filtration $\mathcal{F}_n$ and let $\tau$ be a stopping time (with respect to the same filtration). Suppose that $X_n$ represents your profits (or losses) after $n$ periods. If you decide to quit after $\tau$ periods, then your total profits will be $X_\tau$. In this case your profits after $n$ rounds will in fact be $X_{\tau \wedge n}$. Here $a \wedge b$ denotes the smaller of two numbers $a$ and $b$, i.e.,
\[
a \wedge b = \min(a, b).
\]

**Definition 4.9. (Stopped sequence/process)** We call $X_{\tau \wedge n}$ the sequence **stopped** at $\tau$. It is often denoted by $X_\tau^\tau$. Precisely, for each $\omega \in \Omega$, we have
\[
X_{\tau \wedge n}(\omega) = X_{\tau(\omega)} \wedge n(\omega).
\]

$\square$

**Theorem 4.2. (Stopped process is adapted)** If $X_n$ is a sequence of random variables adapted to a filtration $\mathcal{F}_n$, then so is the sequence $X_{\tau \wedge n}$.

**Proof.** Let $B \subset \mathbb{R}$ be a Borel set. We can write
\[
\{ X_{\tau \wedge n} \in B \} = \{ X_n \in B, \tau > n \} \cup \bigcup_{k=1}^{n} \{ X_k \in B, \tau = k \},
\]
where
\[
\{ X_n \in B, \tau > n \} = \{ X_n \in B \} \cap \{ \tau > n \} \in \mathcal{F}_n
\]
and for each $k = 1, \ldots, n$
\[
\{ X_k \in B, \tau = k \} = \{ X_k \in B \} \cap \{ \tau = k \} \in \mathcal{F}_k \subseteq \mathcal{F}_n.
\]
It follows that for each $n$
\[
\{ X_{\tau \wedge n} \in B \} \in \mathcal{F}_n,
\]
as required.

We know from Proposition 4.1 that it is impossible to turn a fair market into an unfair market, an unfavourable market into a favourable one, or vice versa using any trading strategy. The next proposition shows that this cannot be achieved using a stopping time either (note that stopping is also a trading strategy).

**Proposition 4.3 (Stopped Martingale is a Martingale)** Let $\tau$ be a stopping time.

1. If $X_n$ is a martingale, then so is $X_{\tau \wedge n}$.
2. If $X_n$ is a supermartingale, then so is $X_{\tau \wedge n}$.
3. If $X_n$ is a submartingale, then so is $X_{\tau \wedge n}$.

Proof. This is in fact a consequence of Proposition 4.1. Given a stopping time $\tau$, we put

$$\alpha_n = \begin{cases} 1 & \text{if } \tau \geq n, \\ 0 & \text{if } \tau < n. \end{cases}$$

Notice that $\alpha_n$ is a previsible (that is, $\alpha_n$ is $\mathcal{F}_{n-1}$-measurable). This is because the inverse image $\{ \omega : \alpha_n(\omega) \in B \} = \{ \alpha_n \in B \}$ of any Borel set $B \in \mathcal{B}$ is equal to

$$\begin{align*}
\{ \alpha_n \in B \} &= \begin{cases} \emptyset & \text{if } 0, 1 \notin B \\
\Omega & \text{if } 0, 1 \in B \\
\{ \tau \geq n \} = \{ \tau > n - 1 \} & \text{if } 1 \in B, 0 \notin B , \\
\{ \tau < n \} = \{ \tau \leq n - 1 \} & \text{if } 0 \in B, 1 \notin B,
\end{cases}
\end{align*}$$

and that $\emptyset, \Omega, \{ \tau > n - 1 \}$ and $\{ \tau \leq n - 1 \}$ belongs to $\mathcal{F}_{n-1}$. Setting $X_0 = 0$, we can express $X_{\tau \wedge n}$ as

$$X_{\tau \wedge n} = \alpha_1 (X_1 - X_0) + \cdots + \alpha_n (X_n - X_{n-1}) = \sum_{i=1}^{n} \alpha_i Z_i,$$

where $Z_i = X_i - X_{i-1}$. Therefore, Proposition 4.3 follows from Proposition 4.1 with $Y_n = X_{\tau \wedge n}$.

Example 4.9. (Doubling Strategy) (You could try to beat the system if you had unlimited capital!). Suppose the profit of a unit of investment at each trading day is denoted by $Z_1, Z_2, \ldots$, which take values +1 (up) or -1 (down). You invest $1. If you win (the market goes up), you quit. If you lose (the market goes down), you invest twice as much as last time, and so on until you win. Thus, your trading strategy is

$$\alpha_n = \begin{cases} 2^{n-1} & \text{if } Z_1 = \cdots = Z_{n-1} = -1 \text{ (down)}, \\ 0 & \text{otherwise}. \end{cases}$$

Let $Y_0 = 0$ and

$$Y_n = Z_1 + 2Z_2 + \cdots + 2^{n-1}Z_n,$$

for $n \geq 1$, and consider the stopping time

$$\tau = \min \{ n : Z_n = 1 \text{ (up)} \}.$$
\[ Y_\tau = (Z_1 + 2Z_2 + \cdots + 2^{\tau-2}Z_{\tau-1}) + 2^{\tau-1}Z_{\tau} = -1 - 2 - \cdots - 2^{\tau-2} + 2^{\tau-1} = 1 \]

for any \( \tau \). That is, the profit in the last period is able to cover the lost in the previous \( n - 1 \) periods, with a net gain of $1. However, the expected loss just before the ultimate win is infinite, i.e.,
\[ E(Y_{\tau-1}) = -\infty. \]

To see this, note that the probability that the trading terminates at step \( n \) is
\[ P(\{ \tau = n \}) = P(n-1 \text{ downs followed by an up at step } n) = \frac{1}{2^n}. \]

Therefore,
\[ E(Y_{\tau-1}) = \sum_{n=1}^{\infty} Y_{n-1} P(\{ \tau = n \}) \]
\[ = \sum_{n=2}^{\infty} (-1 - 2 - \cdots - 2^{n-2}) \frac{1}{2^n} \]
\[ = -\sum_{n=2}^{\infty} \frac{2^{n-1} - 1}{2^n} = \sum_{n=2}^{\infty} \frac{1}{2^n} - \sum_{n=2}^{\infty} \frac{1}{2} \]
\[ = \frac{1}{2} - \sum_{n=2}^{\infty} \frac{1}{2^n} = -\infty, \quad (4.5) \]

where the second equality follows from the fact that \( Y_{\tau-1} = -1 - 2 - \cdots - 2^{n-2} \) if \( \tau = n \). The above calculations imply that, you are expected to hold \( \infty \) amount of capital in order to win $1 with the doubling strategy.

### 4.5 Optional Stopping Theorem

If \( X_n \) is a martingale, then it can be shown that (see Exercise 4.10)
\[ E(X_n) = E(X_1) \]

for each \( n \). In Example 4.9, we have \( E(X_{\tau}) = X_{\tau} = 1 \), since the profit \( X_{\tau} \) is always 1 when we stop at \( \tau \). On the other hand, \( E(X_1) = E(Z_1) = 0 \), which is not equal to \( E(X_{\tau}) \). However, if the equality
\[ E(X_{\tau}) = E(X_1) \]
does hold, it can be very useful. The Optional Stopping Theorem provides sufficient conditions for this to hold.
Theorem 4.4. (Optional Stopping Theorem) Let $X_n$ be a martingale and $\tau$ a stopping time with respect to a filtration $\{\mathcal{F}_n\}_{n=1,2,\ldots}$ such that the following conditions hold:

1. $\tau < \infty$ a.s.,
2. $X_\tau$ is integrable,
3. $E(X_n 1_{\{\tau > n\}}) \to 0$ as $n \to \infty$.

Then

$$E(X_\tau) = E(X_1).$$

Proof. Write

$$X_\tau = X_{\tau \wedge n} + (X_\tau - X_n) 1_{\{\tau > n\}},$$

it follows that

$$E(X_\tau) = E(X_{\tau \wedge n}) + E(X_\tau 1_{\{\tau > n\}}) - E(X_n 1_{\{\tau > n\}}).$$  \hspace{1cm} (4.7)

Since $X_{\tau \wedge n}$ is a martingale by Proposition 4.3, the first term on the right-hand side is equal to

$$E(X_{\tau \wedge n}) = E(X_{\tau \wedge 1}) = E(X_1).$$

The last term tends to zero by Assumption 3. Writing $1_{\{\tau > n\}} = \sum_{k=n+1}^{\infty} 1_{\{\tau = k\}}$, the middle term of (4.7) becomes

$$E(X_\tau 1_{\{\tau > n\}}) = \sum_{k=n+1}^{\infty} E(X_k 1_{\{\tau = k\}}),$$

which tends to zero as $n \to \infty$ because the series

$$E(X_\tau) = \sum_{k=1}^{\infty} E(X_k 1_{\{\tau = k\}})$$

is convergent by Assumption 2. It follows that $E(X_\tau) = E(X_1)$, as required. \hfill \Box

Example 4.10. (Expected first hitting time of a random walk) Let $X_n$ be a symmetric random walk as in Example 4.6 and let $K$ be a positive integer. Define the first hitting time (of $\pm K$ by $X_n$) to be

$$\tau = \min\{n : |X_n| = K\}.$$

It can be checked that $\tau$ is a stopping time (see Exercise 4.15). It has been shown in Example 4.6 that $X^2_n - n$ is a martingale. If the Optional Stopping Theorem can be applied, then

$$E(X^2_\tau - \tau) = E(X^2_1 - 1) = 0.$$

This allows us to find the expectation

$$E(\tau) = E(X^2_\tau) = K^2.$$
since $|X_t| = K$.

Next we verify Conditions 1-3 of the Optional Stopping Theorem.

1. First we show that $\mathbb{P}\{\tau = \infty\} = 0$ (i.e., $\tau < \infty$ a.s.). Note that if $X_n \in [-k, k - 1, k - 1]$ and there is a consecutive +1s of length $2K$, the random walk must stop. Therefore, the event $\{\tau > 2Kn\}$ implies that none of the events $\{Z_j = +1, j = 2K(i-1)+1, \ldots, 2Ki\}$, $i = 1, \ldots, n$, could happen. In other words, we have $\{\tau > 2Kn\} \subset \cap_{i=1}^{n} \{Z_j = +1, j = 2K(i-1)+1, \ldots, 2Ki\}$, which implies that

$$
\mathbb{P}\{\tau > 2Kn\} \leq \prod_{i=1}^{n} \left(1 - \frac{1}{2^{2K}}\right) = \left(1 - \frac{1}{2^{2K}}\right)^n \to 0 \quad (4.8)
$$

as $n \to \infty$. Because $\{\tau > 2Kn\}$ for $n = 1, 2, \ldots$ is a contracting sequence of sets (i.e. $\{\tau > 2Kn\} \supset \{\tau > 2K(n+1)\}$), it follows that

$$
\mathbb{P}\{\tau = \infty\} = \mathbb{P}(\bigcap_{n=1}^{\infty} \{\tau > 2Kn\})
= \lim_{n \to \infty} \mathbb{P}\{\tau > 2Kn\} = 0,
$$

completing the argument.

2. We need to show that

$$
E(|X_t^2 - \tau|) < \infty.
$$

Indeed,

$$
E(\tau) = \sum_{n=1}^{\infty} n\mathbb{P}\{\tau = n\} \quad (4.9)
= \sum_{n=0}^{2K} \sum_{k=1}^{2K} (2Kn + k)\mathbb{P}\{\tau = 2Kn + k\}
\leq \sum_{n=0}^{2K} \sum_{k=1}^{2K} 2K(n+1)\mathbb{P}\{\tau > 2Kn\}
\leq 4K^2 \sum_{n=0}^{\infty} (n+1) \left(1 - \frac{1}{2^{2K}}\right)^n
< \infty,
$$

since the series $\sum_{n=1}^{\infty} (n+1)q^n$ is convergent for any $q \in (-1, 1)$. Note that the bound $\mathbb{P}(\{\tau > 2Kn\}) < \left(1 - \frac{1}{2^{2K}}\right)^n$ is established in (4.8). Moreover, $X_t^2 = K^2$, so

$$
E(|X_t^2 - \tau|) \leq E(X_t^2) + E(\tau)
= K^2 + E(\tau)
< \infty.
$$
4.5 Optional Stopping Theorem

3. Since $X_n^2 < K^2$ on $\{\tau > n\}$,

$$E(X_n^2 1_{\{\tau > n\}}) < K^2 P\{\tau > n\} \to 0$$

as $n \to \infty$. Moreover, since $E(\tau) < \infty$, (4.9) implies that $\sum_{j=n}^{\infty} j P(\tau = j) \to 0$ as $n \to \infty$. Thus

$$E(n 1_{\{\tau > n\}}) = \sum_{j=n+1}^{\infty} n P(\tau = j) \leq \sum_{j=n+1}^{\infty} j P(\tau = j) \to 0$$

as $n \to \infty$. It follows that

$$E((X_n^2 - n) 1_{\{\tau > n\}}) \to 0,$$

as required.

**Example 4.11.** For the Doubling Strategy in Example 4.9, we do not have $E(X_{\tau}) = E(X_1)$. On the other hand, the first two conditions of the Optional Stopping Theorem are satisfied. Specifically, Condition 1 is shown in Exercise 4.18; Condition 2 follows from the fact that $X_{\tau} = 1$ and that any constant is integrable. In fact, Condition 3 of the OST is not satisfied for $X_{\tau}$. To see this, note that $E(X_n 1_{\{\tau > n\}}) = (-1 - 2 - \cdots - 2^{n-1}) P(\{\tau > n\}) = (1 - 2^n)2^{-n}$, which does not converge to 0. Thus Condition 3 does not hold.

**Example 4.12.** Let $X_n$ be a symmetric random walk and $\{\mathcal{F}_n\}_{n \geq 1}$ be the filtration defined in Example 4.6. Denote by $\tau$ the smallest $n$ such that $|X_n| = K$ as in Example 4.10. It is standard to verify that

$$Y_n = (-1)^n \cos[\pi(X_n + K)]$$

is a martingale (see Exercise 4.19). We will show that $Y_n$ and $\tau$ satisfy the conditions of the Optional Stopping Theorem and apply the theorem to find $E((-1)^\tau)$.

If Optional Stopping Theorem is applicable, then

$$E(Y_{\tau}) = E(Y_1). \quad (4.10)$$

Since $X_{\tau} = K$ or $-K$, we have

$$Y_{\tau} = (-1)^\tau \cos[\pi(K + X_{\tau})] = (-1)^\tau. \quad (4.11)$$

Together with

$$E(Y_1) = -\frac{1}{2}(\cos[\pi(1 + K)] + \cos[\pi(-1 + K)]) = \cos(\pi K) = (-1)^K. \quad (4.12)$$

Combining (4.10), (4.11) and (4.12), it follows that

$$E((-1)^\tau) = (-1)^K.$$
Thus it remains to verify that \( Y_n \) and \( \tau \) satisfy Conditions 1-3 of the Optional Stopping Theorem. Condition 1 has been verified in Example 4.10. Condition 2 holds since \(|Y_\tau| \leq 1\) implies \( E(|Y_\tau|) \leq 1 < \infty\). To verify condition 3, observe that \(|Y_n| \leq 1\) for all \( n \), so

\[
E(Y_n 1_{\{\tau > n\}}) \leq E(Y_n 1) \leq E(1_{\{\tau > n\}}) = P(\{\tau > n\}).
\]

Thus, we have from (4.8) that

\[
\lim_{n \to \infty} |E(Y_n 1_{\{\tau > n\}})| \leq \lim_{n \to \infty} P(\tau > n) = P(\tau = \infty) = 0,
\]

and the proof is completed.

**Example 4.13. (Asymmetric Simple Random Walk)** The asymmetric simple random walk is defined by

\[ X_n = Z_1 + \cdots + Z_n, \]

where \( P(Z_i = 1) = p \) and \( P(Z_i = -1) = 1 - p \) for all \( i \). W.L.O.G. assume that \( p > 1/2 \). Let \( \tau_x = \inf\{n : X_n = x\} \) and \( T = \min(\tau_a, \tau_b) \), \( a < 0 < b \) be the first time of hitting the boundary of the interval \((a, b)\). Define

\[ \varphi(x) = \left((1 - p)/p\right)^x. \]

We have the following results by the OST. We only illustrate the application of OST and the verification of conditions is left to Exercise 4.22.

1. \[ P(\tau_a < \tau_b) = \frac{\varphi(b) - 1}{\varphi(b) - \varphi(a)}. \] (4.13)

2. If \( a < 0 \), then \( P(\tau_a < \infty) = \varphi(-a) \).

3. If \( b > 0 \), then \( E(\tau_b) = b/(2p - 1) \).

1. First, it can be checked that \( \varphi(X_n) \) is a martingale (Exercise 4.22). Using OST on \( \varphi(X_T) \), we have

\[
1 = \varphi(0) = E(\varphi(X_T)) = P(\tau_a < \tau_b) \varphi(a) + P(\tau_b < \tau_a) \varphi(b). \] (4.14)

Together with \( P(\tau_a < \tau_b) + P(\tau_b < \tau_a) = 1 \), solving for \( P(\tau_a < \tau_b) \) gives (4.13).

2. It follows from taking \( b \to \infty \) in (4.13) and noting that \( \varphi(b) \to 0 \) as \( b \to \infty \).

3. Applying OST to \( X_{\tau_b} - (2p - 1)\tau_b \) (see Exercise 4.22 for the fact that \( X_n - (2p - 1)n \) is a martingale), we have

\[
b - (2p - 1)E(\tau_b) = E(X_{\tau_b} - (2p - 1)\tau_b) = E(X_1 - (2p - 1)) = 0,
\]

i.e., \( E(\tau_b) = b/(2p - 1) \).
Remark 4.5. (A useful trick) To verify OST in Assertion 3 of Example 4.13, instead of showing that $E(X_n 1_{\{\tau_b > n\}}) \to 0$, we can argue that

$$E(X_{\tau_b \wedge n} - (2p - 1)(\tau_b \wedge n)) = E(X_1 - (2p - 1)) = 0.$$ 

This is justified as the stopped martingale $X_{\tau_b \wedge n}$ is a martingale (Proposition 4.3). Next, taking $n \to \infty$ and applying Dominant Convergence Theorem and Monotone Convergence Theorem, respectively, we get $E(X_{\tau_b \wedge n}) \to E(X_{\tau_b}) = b$ and $E(\tau_b \wedge n) \to E(\tau_b)$. Thus Assertion 3 follows.

4.6 Martingale Convergence Theorem

Definition 4.10. (Upcrossing Strategy) Given an adapted sequence of r.v. $X_1, X_2, \ldots$, and two real numbers $a < b$, the upcrossing strategy $\alpha_1, \alpha_2, \ldots$ is given by

$$\alpha_1 = 0, \quad \alpha_{n+1} = \begin{cases} 
1 & \text{if } \alpha_n = 0 \text{ and } X_n < a \\
1 & \text{if } \alpha_n = 1 \text{ and } X_n \leq b \\
0 & \text{otherwise}
\end{cases} \quad (4.15)$$

for $n = 1, 2, \ldots$. When $\alpha_k = 1$ and $\alpha_{k+1} = 0$, we say that there is an upcrossing at time $k$. Let the upcrossing sequence $u_1, u_2, \ldots$ be the time indexes where upcrossings occur. The number of upcrossings made up to time $n$ is defined by

$$U_n[a, b] = \max\{k : u_k \leq n\}.$$ 

Remark 4.6.

The upcrossing strategy is related to the trading principle of buy-low-sell-high in finance. Imagine

- $X_n$ is the stock price at time $n$.
- $\alpha_n = 1$ means holding one unit of the stock, 0 otherwise.

The upcrossing strategy means that, when the stock price $X$ falls below $a$, we hold the stock. We wait until $X$ reaches $b$, where $b > a$, and sell the stock. Then we wait for the next chance that $X$ falls below $a$, and so forth. The upcrossing corresponds to selling the stock and completing a cycle of trading. Note that each upcrossing increases the total wealth by at least $b - a$.

Theorem 4.5. The upcrossing strategy $\{\alpha_n\}$ in Definition 4.10 is previsible.

Proof. We need to prove that $\alpha_n$ is $\mathcal{F}_{n-1}$ measurable for each $n$. We will prove by mathematical induction. First, note that by definition $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and the constant $\alpha_1 = 0$ is $\mathcal{F}_0$ measurable since for any Borel set $B \in \mathcal{B}$, we have

$$\{\alpha_1 \in B\} = \begin{cases} 
\Omega & \text{if } 0 \in B \\
\emptyset & \text{if } 0 \notin B
\end{cases}.$$
Next, suppose that \( \alpha_k \) is \( \mathcal{F}_{k-1} \) measurable, we have for any Borel set \( B \in \mathcal{B} \) that

\[
\{ \alpha_{k+1} \in B \} = \begin{cases} 
\mathcal{F} & \text{if } \{ 0,1 \} \cap B \\
\{ \alpha_k = 0 \} \cap \{ X_k < a \} \cup \{ \alpha_k = 1 \} \cap \{ X_k \leq b \} & \text{if } 1 \in B, 0 \notin B \\
\text{the complement of above} & \text{if } 1 \notin B, 0 \in B \\
\emptyset & \text{if } 0 \notin B, 1 \in B.
\end{cases}
\]

Since \( \{ \alpha_k = 0 \}, \{ \alpha_k = 1 \}, \{ X_k < a \} \) and \( \{ X_k \leq b \} \) are \( \mathcal{F}_k \) measurable by definition, it follows that \( \alpha_{k+1} \) is \( \mathcal{F}_k \) measurable.

**Theorem 4.6. (Upcrossings Inequality)** Let \( \{ X_n \}_{n=1}^{\infty} \) be a supermartingale and \( a < b \), then

\[
(b-a)E(U_n[a,b]) \leq E((X_n-a)^-),
\]

where \( x^- = \max(0,-x) \).

**Proof.** Let \( X_0 = 0 \) and

\[
Y_n = \alpha_1(X_1 - X_0) + \cdots + \alpha_n(X_n - X_{n-1}),
\]

be the total profits at time \( n \). Note from Proposition 4.1 that \( Y_n \) is a supermartingale.

Suppose that \( U_n[a,b] = k \) at time \( n \), with \( 0 < u_1 < \cdots < u_k < n \). Since each upcrossing increases the total profits by \( b-a \), we have

\[
Y_{u_i} - Y_{u_{i-1}} \geq b-a,
\]

for \( i = 1, \ldots, k \), where \( u_0 \equiv 0 \). At time \( n \), if \( X_n \geq a \), then \( Y_n - Y_{u_k} \geq 0 = -(X_n - a)^- \). On the other hand, if \( X_n < a \), then \( Y_n - Y_{u_k} \geq X_n - a = -(X_n - a)^- \). Putting together, we have

\[
Y_n - Y_{u_k} \geq -(X_n - a)^-.
\]

Summing over (4.18) and (4.17) for \( i = 1, \ldots, k \), we have

\[
Y_n \geq (b-a)U_n[a,b] - (X_n-a)^-.
\]

Taking expectation both sides, we get

\[
E(Y_n) \geq (b-a)E(U_n[a,b]) - E((X_n-a)^-).
\]

Lastly, since \( Y_n \) is a supermartingale, we have \( 0 \geq E(Y_1) \geq E(Y_n) \), thus

\[
E((X_n-a)^-) \geq (b-a)E(U_n[a,b]),
\]

completing the proof.

**Example 4.14.** For a random walk \( X_n = Z_1 + \cdots + Z_n \), \( Z_i \overset{d}{\sim} N(0,1) \). We have \( X_n \overset{d}{\sim} N(0,n) \). Suppose that the upcrossing strategy is performed with \( a = -1 \) and \( b = 1 \).
4.6 Martingale Convergence Theorem

Using (4.16) the expected number of upcrossing by time \( n \) satisfies

\[
(1 - (-1))E(U_n[-1, 1]) \leq \int (x - (-1))^{-1} \phi(x, 0, n) \, dx
\]

\[
\Leftrightarrow E(U_n[-1, 1]) \leq -\frac{1}{2} \int_{-\infty}^{-1} (x + 1) \phi(x, 0, n) \, dx
\]

\[
= -\frac{1}{2} \left( \int_{-\infty}^{-1} xe^{-x^2/2n}/\sqrt{2\pi n} \, dx + \int_{-1}^{-1} e^{-x^2/2n}/\sqrt{2\pi n} \, dx \right)
\]

\[
= \frac{\sqrt{ne^{-\frac{1}{4}}}}{2\sqrt{2\pi}} - \frac{1}{2} \Phi(-\frac{1}{\sqrt{n}}),
\]

where \( \phi(x, 0, n) = e^{-x^2/2n}/\sqrt{2\pi n} \) is the p.d.f. of the Normal random variable with mean 0 and variance \( n \) and \( \Phi \) is the c.d.f. of the standard Normal random variable.

**Theorem 4.7. (Doob’s Martingale Convergence Theorem)** Suppose that \( \{X_n\}_{n=1,\ldots} \) are supermartingale with respective to the filtration \( \{\mathcal{F}_n\}_{n=1,\ldots} \), such that

\[
\sup_n E(||X_n||) = M < \infty.
\]

There is an integrable random variable \( X \) such that

\[
\lim_{n \to \infty} X_n = X, \quad a.s.
\]

**Remark 4.7.** Note that this theorem is valid for martingale and submartingale since

- every martingale is a supermartingale.
- \( -X_n \) is a supermartingale if \( X_n \) is a submartingale.

**Proof.** (Theorem 4.7) By the Upcrossing Inequality, for every \( -\infty < a < b < \infty \),

\[
E(U_n[a, b]) \leq \frac{E((X_n-a)^-)_{b-a}}{b-a} \leq \frac{M + |a|}{b-a} < \infty.
\]

Since \( U_n[a, b] \) is non-decreasing in \( n \), the MCT implies

\[
E(\lim_{n \to \infty} U_n[a, b]) = \lim_{n \to \infty} E(U_n[a, b]) \leq \frac{M + |a|}{b-a} < \infty.
\]

This implies that

\[
P(\lim_{n \to \infty} U_n[a, b] < \infty) = 1,
\]

otherwise the expectation must be infinity. To show that \( X_n \overset{a.s.}{\rightarrow} X \), we need to show that

\[
P(\omega : \liminf_n X_n(\omega) = \limsup_n X_n(\omega)) = 1.
\]
It is equivalent to showing that \( \mathbb{P}(B) = 0 \) where
\[
B = \{ \liminf_{n} X_{n} < \limsup_{n} X_{n} \} \subset \Omega. \tag{4.22}
\]
Note that if \( \omega \in B \) then
\[
\liminf_{n} X_{n}(\omega) < a_{\omega} < b_{\omega} < \limsup_{n} X_{n}(\omega), \tag{4.23}
\]
for some rational \( a_{\omega} \) and \( b_{\omega} \). However, (4.22) implies that \( \{X_{n}\} \) has upcrossing from \( a_{\omega} \) to \( b_{\omega} \) infinity many times, i.e., \( \lim_{n \to \infty} U_{n}[a_{\omega}, b_{\omega}](\omega) = \infty \). Thus we have
\[
B \in \bigcup_{a, b \in \mathbb{Q}} \{ \lim_{n \to \infty} U_{n}[a, b] = \infty \}.
\]
Therefore, we have from (4.20) that
\[
\mathbb{P}(B) \leq \sum_{a, b \in \mathbb{Q}} \mathbb{P}(\lim_{n \to \infty} U_{n}[a, b] = \infty) = 0.
\]
Note the importance of introducing the countable rational numbers \( a_{\omega}, b_{\omega} \): an uncountable sum of zeros may not be zero.

It remains to show that the limit \( X \) is an integrable random variable. By Fatou’s lemma,
\[
E(|X|) = E(\liminf_{n} |X_{n}|) = E(\lim_{n} |X_{n}|) \quad \text{(since the limit exists)}
\]
\[\leq \liminf_{n} E(|X_{n}|) \quad \text{(Fatou’s lemma)}
\]
\[< \sup_{n} E(|X_{n}|) = M < \infty, \]
completing the proof.

**Example 4.15.** Let \( Z_{1}, Z_{2}, \ldots \) be i.i.d. random variables with \( E(Z_{m}) = 1 \) and \( \mathbb{P}(Z_{m} = 1) < 1 \). It is easy to verify that \( X_{n} = \prod_{m \leq n} Z_{m} \) is a martingale and \( E(X_{n}) = 1 \) for all \( n \).

Therefore, Theorem 4.7 implies that \( X_{n} \xrightarrow{a.s.} X \) for some r.v. \( X \). Note that for positive \( A \) and \( B \), if \( A > \delta \) and \( B > \epsilon \), then \( AB > \delta \epsilon \). Speaking in set theory, we have
\[
\{A > \delta\} \cap \{B > \epsilon\} \subset \{AB > \delta \epsilon\}, \quad \text{or}
\]
\[
\mathbb{P}(AB > \delta \epsilon) \geq \mathbb{P}(\{A > \delta\} \cap \{B > \epsilon\}). \tag{4.24}
\]

Now, putting \( A = X_{n} \) and \( B = |Z_{n+1} - 1| \), we have
\[
\mathbb{P}(|X_{n+1} - X_{n}| > \delta \epsilon) = \mathbb{P}(X_{n} | Z_{n+1} - 1 > \delta \epsilon)
\]
\[\geq \mathbb{P}(\{X_{n} > \delta\} \cap \{|Z_{n+1} - 1| > \epsilon\}) \quad \text{(by (4.24))}
\]
\[= \mathbb{P}(X_{n} > \delta) \mathbb{P}(|Z_{n+1} - 1| > \epsilon) \quad \text{(by independence)}
\]

For any \( \delta > 0 \) and \( \epsilon > 0 \), the L.H.S. of the above converges to 0 by the existence of limit of \( X_{n} \). For R.H.S, \( \mathbb{P}(|Z_{n+1} - 1| > \epsilon) \) is strictly positive for some \( \epsilon > 0 \).
4.7 Exercises

4.8 Exercises

Since \( P(Z_m = 1) < 1 \) in the set-up. Therefore, \( P(X_n > \delta) \to 0 \) for any \( \delta > 0 \). As by construction, \( X_n \) is positive, we conclude that

\[ X_n \overset{a.s.}{\to} 0. \]

It is surprising to see that \( E(X_n) = 1 \) for all \( n \) but \( X_n \overset{a.s.}{\to} 0. \)

4.7 Exercises

Exercise 4.8 Let \( X_1, X_2, \ldots \) be a sequence of coin tosses and let \( \mathcal{F}_n \) be the \( \sigma \)-field generated by \( X_1, \ldots, X_n \). Write down the all the elements in each of \( \mathcal{F}_i \), \( i = 1, 2, 3 \).

Exercise 4.9 Let \( X_1, X_2, \ldots \) be a sequence of coin tosses and let \( \mathcal{F}_n \) be the \( \sigma \)-field generated by \( X_1, \ldots, X_n \). For each of the following events find the smallest \( n \) such that the event belongs to \( \mathcal{F}_n \):

- \( A = \{ \text{the first occurrence of heads is preceded by no more than 10 tails} \} \)
- \( B = \{ \text{there is at least 1 head in the sequence } X_1, X_2, \ldots \} \)
- \( C = \{ \text{the first 100 tosses produce the same outcome} \} \)
- \( D = \{ \text{there are no more than 2 heads and 2 tails among the first 5 tosses} \} \)

Exercise 4.10 Show that if \( X_n \) is a martingale with respect to \( \mathcal{F}_n \), then

\[ E(X_1) = E(X_2) = \ldots. \]

Hint: What is the expectation of \( E(X_{n+1} | \mathcal{F}_n) \)?

Exercise 4.11 Suppose that \( X_n \) is a martingale with respect to a filtration \( \mathcal{F}_n \). Show that \( X_n \) is a martingale with respect to the filtration

\[ \mathcal{G}_n = \sigma(X_1, \ldots, X_n). \]

Hint: Observe that \( \mathcal{G}_n \subset \mathcal{F}_n \) and use the tower property of conditional expectation.

Exercise 4.12 Let \( X_n \) be a symmetric random walk and \( \{ \mathcal{F}_n \}_{n \geq 1} \) be the filtration defined in Example 4.6. Show that

\[ Y_n = (-1)^n \cos(\pi X_n) \]

is a martingale with respect to \( \mathcal{F}_n \).

Hint: Try to manipulate \( E((-1)^{n+1} \cos(\pi X_{n+1}) | \mathcal{F}_n) \) to obtain \( (-1)^n \cos(\pi X_n) \). Use a similar argument as in Example 4.6 to achieve this. But, first of all, make sure that \( Y_n \) is integrable and adapted to \( \{ \mathcal{F}_n \}_{n \geq 1} \).

Exercise 4.13 Recall Example 4.6. Verify the measurability and integrability of the martingale \( Y_n^2 - n \).
Exercise 4.14 Show that the following conditions are equivalent:

1. \( \{ \tau \leq n \} \in \mathcal{F}_n \) for each \( n = 1, 2, \ldots \)
2. \( \{ \tau = n \} \in \mathcal{F}_n \) for each \( n = 1, 2, \ldots \)

Hint: Express \( \{ \tau \leq n \} \) in terms of the events \( \{ \tau = k \} \), where \( k = 1, \ldots, n \). Express \( \{ \tau = n \} \) in terms of the events \( \{ \tau \leq k \} \), where \( k = 1, \ldots, n \).

Exercise 4.15 Let \( X_n \) be a sequence of random variables adapted to a filtration \( \mathcal{F}_n \) and let \( B \subseteq \mathbb{R} \) be a Borel set. Show that the time of first entry of \( X_n \) into \( B \),

\[ \tau = \min \{ n : X_n \in B \}, \]

is a stopping time.

Hint: Example 4.8 covers the case when \( B = (-\infty, 0] \cup [10, \infty) \). Extend the argument to an arbitrary Borel set \( B \).

Exercise 4.16 Show that, if \( X_n \) is a martingale w.r.t. \( \mathcal{F}_n \), then is \( X_3^n \) a martingale? If \( X_n = \sum_{i=1}^n Z_i \), where \( Z_i = 1 \) or \( -1 \) with equal probabilities, can you construct a martingale involving \( X_3^n \)?

Exercise 4.17 Verify that the stopped process \( X_{\tau \wedge n} \) in Example 4.9 is a martingale. Show also that \( \mathbb{P}(\tau < \infty) = 1 \).

Exercise 4.18 Find the expected length of a game, \( E(\tau) \), using the doubling strategy in Example 4.9.

Exercise 4.19 In Example 4.12, show that \( Y_n \) is a martingale.

Exercise 4.20 Show that a previsible martingale is a constant.

Exercise 4.21 Let \( X_n \) be the symmetric random walk defined in Example 4.6. Let \( Y_n = X_n + 1 \) be a random walk started from 1 at time 0. Let \( \tau = \inf \{ n : Y_n = 0 \} \).

- Show that \( \tau \) is a stopping time.
- Find \( E(\tau) \)
- Show that \( Y_n \) and \( Y_{\tau \wedge n} \) are martingales.
- Is \( E(Y_\tau) = E(Y_1) \)? Explain by verifying the Optional Stopping Theorem.

Exercise 4.22 In Example 4.13, verifying that \( \varphi(X_n) \) and \( X_n - (2p - 1)n \) are martingales. Justify the applications of OST to the two martingales.

Exercise 4.23 Fill in the details of the applications of DCT and MCT in Remark 4.5.

Exercise 4.24 In the upcrossing strategy, if \( \{ X_n \}_{n=1}^{14} = (3, 4, 6, 1, 3, 4, 6, 7, 2, 1, 3, 8, 3, 3) \), write down the paths \( \{ \alpha_n \} \) and the final total profit for the cases: i) \((a, b) = (3.1, 3.9)\), ii) \((a, b) = (2.9, 4.1)\).
Exercise 4.25 For a drifted random walk \( X_n = Z_1 + \ldots + Z_n \), \( Z_i \sim N(\mu, 1) \). We have \( X_n \sim N(n\mu, n) \). Suppose that the upcrossing strategy is performed with \( a = -1 \) and \( b = 1 \). In terms of \( \mu \) and \( n \), find an upper bound for the expected number of upcrossing by time \( n \). Give the values for \( \mu = 0, -1, -2 \) and \( n = 10, 100, 1000 \) (9 combinations).

Exercise 4.26 If \( M_t^{(1)} \) and \( M_t^{(2)} \) are martingales, then is \( Y_t = M_t^{(1)} M_t^{(2)} \) a martingale?