1. By Theorem 3.2 in textbook, the stationary condition for $\phi_1$ is $|\phi_1| < 1$.

2. Similar to 1, the conditions is $|r_1| < 1$ and $|r_2| < 1$.

3.(a) the characteristic function is $\phi(x) = 1 - \alpha x - \beta x^2$. Since the roots of the equation $x_1$, $x_2$ have relation $|x_1 x_2| = \frac{1}{|\beta|} > 1$, so $|\beta| < 1$. Next we consider two situations:
First, $\beta > 0$ so for $\phi(x)$, since the two roots are greater than 1 in absolute value. So
$$\phi(1) > 0 \quad \phi(-1) > 0$$
from the above equation, we have
$$\beta < 1 - \alpha, \beta < 1 + \alpha$$
we argue that the characteristic equation roots $x_1$, $x_2$ are not simultaneously larger than 1. Since if $x_1 x_2 = -\frac{1}{\beta} > 1$, we have $\beta < 1$, this is a contradiction. So the two are one larger than 1 and the other is smaller than -1.

(b) From $Z_t = \alpha Z_{t-1} + \beta Z_{t-2} + a_t$, we have
$$\gamma(0) = \alpha \gamma(1) + \beta \gamma(2) + \sigma_a^2$$
so
$$1 = \alpha \rho_1 + \beta \rho_2 + \frac{1}{2}$$

4.(a) ARMA
(b)
(i) To let $Z_t$ be stationary, for the characteristic equation of AR process
\( \phi(x) = 1 - \theta x \) and MA process \( \theta(x) = 1 + \alpha x + \beta x^2 \), the absolute value of the roots are all lies out of the unit circle.

(ii) to let \( Z_t \) be causal iff \( \phi(x) \neq 0 \) for \( |x| \leq 1 \).

(iii) to let \( Z_t \) be invertible iff \( \theta(x) \neq 0 \) for \( |x| \leq 1 \).

(c)
To get the autocorrelation function of \( Z_t \), we first express \( Z_t \) by infinite sum.
\[
\phi(B) Z_t = \theta(B) a_t
\]
where \( B \) is the backshiftor. So express \( Z_t \) in terms of \( a_t \)
\[
Z_t = \frac{\theta(B)}{\phi(B)} a_t = \sum_{j=0}^{\infty} \psi_j a_{t-j}
\]
\[
= (1 + \alpha B + \beta B^2)(1 + \theta B + \theta^2 B^2 + \theta^3 B^3 + \theta^4 B^4 + \cdots) a_t
\]
\[
= (1 + (\theta + \alpha) B + (\theta^2 + \alpha \theta + \beta) B^2 + \theta(\theta^2 + \alpha \theta + \beta) B^3 + \theta^2(\theta^2 + \alpha \theta + \beta) B^4) a_t
\]
so
\[
Var(Z_t) = (1 + (\theta + \alpha)^2 + (\theta^2 + \alpha \theta + \beta)^2 + \theta^2(\theta^2 + \alpha \theta + \beta)^2 + \cdots) \sigma_a^2
\]
\[
= \sigma_a^2 + (\theta + \alpha)^2 \sigma_a^2 + \frac{\theta^2(\theta^2 + \alpha \theta + \beta)^2}{1 - \theta^2} \sigma_a^2
\]
For the autocovariance function, \( k > 0 \)
\[
\gamma(k) = \sigma_a^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|}
\]
where
\[
\begin{align*}
\psi_0 &= 1 \\
\psi_1 &= \theta + \alpha \\
\psi_2 &= \theta^2 + \alpha \theta + \beta \\
\psi_3 &= \theta(\theta^2 + \alpha \theta + \beta) \\
\psi_4 &= \theta^2(\theta^2 + \alpha \theta + \beta) \\
&\vdots \quad \vdots \\
\psi_n &= \theta^{n-2}(\theta^2 + \alpha \theta + \beta)
\end{align*}
\]
so we can get autocorrelation function by $\rho_k = \gamma(k) / \gamma(0)$. 