Abstract. This paper provides sufficient conditions for the enlargement of filtrations on Poisson space, making use of a calculus of variations whose gradient is with respect to jump times. It is found that the construction of a measure-valued Malliavin calculus in Imkeller et al. (2001) directly applies analogously to the Poisson space with the Malliavin type calculus defined in Mensi and Privault (2003). Some new, explicit calculations of enlargement of filtrations are provided, making use of the translation operator defined in Nualart and Vives (1990).

1. Introduction

Most models of financial markets assume that traders’ knowledge is adapted to the natural filtration generated by the assets’ price processes. In other words, the prices from the beginning of the trading interval until now entirely constitute what they know about the market. Any trading strategy of theirs must also therefore depend on past information. But what if there is a trader with an extra piece of information that is not known to the public until the end of the trading interval? How can he take advantage of his privileged position?

We call such a fortunate trader an ‘insider’ and his extra information is represented by a random variable \( L \) that is only revealed to the other, regular traders at the end of the trading interval. The first approach to studying the effects of such asymmetrical information in the familiar context of martingale theory was taken in Pikovsky and Karatzas (1996). In that paper the authors sought to maximize the expected logarithmic utility from the final value of portfolios (consisting of \( d \) risky stocks generated by a \( d \)-dimensional Brownian motion and one non-risky asset) owned by regular traders and the insider. Taking the trading interval to be \([0, 1]\), the information flow as time progresses for the regular trader is represented by \( \{F_t\}_{0 \leq t \leq 1} \), the completed natural filtration generated by the underlying Brownian motion \( W \). The \( F_1 \)-measurable random variable \( L \) clearly enriches the information flow of the insider. Indeed, the insider’s knowledge is represented by an enlarged filtration \( \{G_t\}_{0 \leq t \leq 1} \), where \( G_t := F_t \vee \sigma(L) \) for \( t \in [0, 1] \).

Pikovsky and Karatzas (1996) proved that for many random variables \( L \) the insider can have infinite expected logarithmic utility from his terminal wealth. This is true when \( L = W_1 \), for example, where \( W_1 \) is the value of the generating Brownian motion at the end of the trading interval. An example where the insider’s utility remains finite is \( L = \lambda W_1 + (1 - \lambda)\epsilon \) where \( 0 < \lambda < 1 \) and \( \epsilon \) is a standard normal random variable, independent of the Brownian motion \( W \). The paper considered
several different choices for \( L \) and noted that crucial to the calculation of the expected logarithmic utility to the insider in each case was a \( \mathcal{G}_t \)-measurable random process, later to be called the ‘information drift’ in Imkeller (2003), that can be subtracted from the \( \mathcal{F}_t \)-Brownian motion \( W \) to create a \( \mathcal{G}_t \)-Brownian motion \( \tilde{W} \). Should such a process, which we will label \( \mu^L_s \), exist then Pikovsky and Karatzas (1996) showed that the additional expected logarithmic utility of \( L \) to the insider is
\[
\frac{1}{2} \mathbb{E} \left[ \int_0^1 (\mu^L_s)^2 ds \right].
\]
It is clear that if \( \mu^L_s \) is infinite on a set of probability measure greater than zero, then the advantage to the insider is in theory infinite.

Further investigation into the relationship between additional information \( L \) and the information drift \( \mu^L_s \) began with an assumption used in the very important paper by Jacod (1985), known eponymously in the literature as ‘Jacod’s condition’:

*The regular conditional distributions of \( L \) given \( \mathcal{F}_t \) are absolutely continuous with respect to the law of \( L \) for all \( t \in [0, 1) \).*

Jacod (1985) shows that under this assumption, there exists an information drift that when subtracted from \( \mathcal{F}_t \)-Brownian motion \( W \) creates a \( \mathcal{G}_t \)-Brownian motion \( \tilde{W} \). Moreover, the information drift is given explicitly.

In fact, Jacod (1985) works in the more general setting of semimartingale theory and the relevant result for the Brownian motion setting is a corollary of Théorème 2.1 of those lecture notes. Amendinger et al. (1998) showed how to calculate the additional expected logarithmic utility in the semimartingale setting, with the interesting result that it has a simple representation in terms of the entropy of \( L \).

Jacod’s condition implies that a \( \mathcal{G}_t \)-Brownian motion \( \tilde{W} \) can be represented as
\[
W_t = \tilde{W}_t + \int_0^t \mu^L_s ds
\]
allowing not only the information drift but also the additional expected logarithmic utility to be found. However, it is not necessary for such an integral representation to exist. Imkeller et al. (2001) considered the important example of \( L = \sup_{0 \leq t \leq 1} W_t \) and showed that it did not satisfy Jacod’s condition. Motivated by this example, the authors generalized Jacod’s condition in the Wiener setting by first showing that the conditional law of \( L \) given \( \mathcal{F}_t \) \((P(L \in dx|\mathcal{F}_t))\) has an integral representation and that crucially this can be found explicitly. The method that achieves this uses a natural application of Malliavin calculus. To explain heuristically, the Itô representation theorem tells us that for a square-integrable \( F \) on the Wiener probability space, there exists a process \( \phi_s \) such that
\[
F = \mathbb{E}[F] + \int_0^1 \phi_s dW_s
\]
but the theorem does not tell us what \( \phi_s \) is. Fortunately the Clark-Ocone formula does, for suitable \( F \):
\[
F = \mathbb{E}[F] + \int_0^1 \mathbb{E}[D_s F|\mathcal{F}_s] dW_s
\]
and this is where Malliavin calculus enters the picture, through the Malliavin gradient \( D \). By telling us the integral representation of \( P(L \in dx|\mathcal{F}_t) \) explicitly, via a Clark-Ocone formula, Imkeller et al. (2001) find a new, simple expression for the information drift \( \mu^L_s \) and consequently derive new results for its behaviour. However, \( P(L \in dx|\mathcal{F}_t) \) is not a simple random variable but a measure-valued random variable and in order to make sense of an integral representation of \( P(L \in dx|\mathcal{F}_t) \),
Imkeller et al. (2001) create a measure-valued Malliavin calculus with its own, measure-valued Clark-Ocone formula.

Most of the literature that considers enlarged filtrations in a financial setting use Brownian motion as the generating random process, as in Pikovsky and Karatzas (1996), Imkeller et al. (2001) and Imkeller (2003). Indeed, Elliott et al. (1997) examined the case when the stock price process is generated by two independent Brownian motions and the insider has privileged information about one of them. The first paper to consider a price model with jumps in this context was Elliott and Jeanblanc (1999), wherein the authors modeled the logarithm of the asset price by a Brownian motion with simple Poisson jump process. The existence of arbitrage opportunities for an insider with various examples of privileged knowledge was investigated via the non-existence of equivalent martingale measures, with the central example being $L = N_T$, i.e. the insider knows the number of jumps that will occur in the trading interval.

An early work which considers (and derives a sufficient condition for) enlargement of filtration on Poisson space is Pontier (2000), which constructs a Dirichlet form on a Poisson space, while Mensi and Privault (2003) were the first to consider an enlargement of filtration on the Poisson space with financial applications. Their leading example is a financial market where the logarithmic price is the sum of a pure jump process with drift and an insider who knows the interarrival time between the jumps straddling the end of the trading interval. Much of Mensi and Privault (2003) is devoted to finding sufficient conditions for a random variable on the Poisson space to have a conditional density, i.e. their conditional distribution is continuous with respect to the Lebesgue measure. These proofs make use of ‘Chaotic and variational calculus in discrete and continuous time for the Poisson process’ (Privault (1994)), which we shall call for reasons of brevity and similarity the Poisson Malliavin calculus.

With the existence of a Malliavin type calculus on the Poisson space and recalling the approach in Imkeller et al. (2001) to investigating enlargement of filtrations, it is natural to ask whether Privault’s Poisson Malliavin calculus can be used in a similar fashion to generalize Jacod’s condition on the Poisson space. There is some work in this direction in Mensi and Privault (2003), making use of a Clark-Ocone formula to provide sufficient conditions for the existence of an information drift.

It is the aim of this paper to generalize and further the work began in Pontier (2000) and Mensi and Privault (2003) through the method shown in Imkeller et al. (2001): construct a measure-valued Poisson Malliavin calculus to make sense of and provide sufficient conditions for an integral representation of conditional laws and use this to generalize Jacod’s condition for the existence of an information drift on the Poisson space.

The rest of the paper is structured as follows: Section 2 introduces the version of Poisson Malliavin calculus that will be used; Section 3 defines the measure-valued Poisson Malliavin calculus and culminates with a Clark-Ocone formula; Section 3 uses this formula to provide sufficient conditions for the existence of an information drift; Section 4 illustrates applications with some notable examples and the final section concludes.
2. A Poisson Malliavin calculus

The Malliavin calculus is an infinite-dimensional differential calculus on the Wiener space. For a gentle and intuitive introduction to Malliavin calculus (also known as the stochastic calculus of variations), we recommend the lectures of Øksendal (1997), Imkeller (2008), and Malliavin and Thalmaier (2010); and for a more thorough study of its main features and applications to finance, we recommend Nualart (2006).

Carlen and Pardoux (1990) defined a gradient operator for random variables on the Poisson space by shifting jump times and in doing so derived results analogous to those in the Wiener space-based Malliavin calculus; namely the authors found an integration by parts formula and conditions for the law of a random variable to be absolutely continuous with respect to the Lebesgue measure. Privault (1994) developed a stochastic calculus of variations for the Poisson space using a chaotic decompositions of a given random variable. Again, interesting parallels to the Malliavin calculus on the Wiener space were found. Over the Poisson space, Laguerre polynomials play the role of Hermite polynomials on the Wiener space as a basis for the chaos decomposition (see respectively Prop. 4 in Privault (1994) and Thm, 1.1.2 in Nualart (2006)). Privault (1994) defines a gradient operator that is similar to that in Carlen and Pardoux (1990) and proves that it satisfies an integration by parts formula, a chain rule and, crucially for our purposes, a Clark-Ocone type formula (Thm. 1 Privault (1994)).

Just as the Malliavin calculus has applications to finance, so does its Poisson space-based cousin. Following a method started in the Wiener space setting in Fournié et al. (1999) and Fournié et al. (2001), using Malliavin calculus to estimate sensitivities of financial derivatives faster, El Khatib and Privault (2004) use a Poisson version to compute Greeks for Asian options in a market with jumps. In a similar fashion, Privault and Wei (2004) computes sensitivities of ruin probabilities for insurance portfolios. As mentioned before, Mensi and Privault (2003) touch upon insider trading in a market where the logarithmic asset price follows a jump process.

The fact that each of the papers by Privault uses a slightly different gradient operator in constructing their Malliavin calculus on Poisson space is testament to the flexibility of this approach of addressing problems in finance. For example, El Khatib and Privault (2004) and Privault and Wei (2004) both incorporate weight functions into their defined gradient that can be chosen to reduce the variance of sensitivity estimates. The gradient defined in Privault (1994) is given in terms of partial derivatives of a Poisson space random variable with respect to interarrival times.

In this paper we shall use the ‘damped gradient’ defined in Mensi and Privault (2003) because, as the authors noted,

> of its particular properties that make it closer to the Wiener space derivative.

Since we are following the lead of the Wiener-space based results of Imkeller et al. (2001), it makes sense to use the damped gradient for the standard Poisson process. We summarize this Poisson Malliavin calculus here.

The Poisson space \((\mathcal{B}, \mathcal{F}, \mathbb{P})\) consists of \(\mathcal{B}\), a sequence space, \(\mathbb{P}\), a probability measure under which the canonical projections \(\tau_k : B \to \mathbb{R}\) form a sequence of i.i.d.
exponentially distributed random variables. From these \(\tau_k\) (which we intuitively treat as interarrival times) we can construct familiar random variables like jump times \(T_k\) and the Poisson process \(N_t\). To wit, the \(n\)th jump time of the Poisson process, \(T_n\), is defined by
\[
T_n := \sum_{k=0}^{n-1} \tau_k
\]
and Poisson process \(N_t\) by
\[
N_t(\omega) := \sum_{k=1}^{\infty} 1_{[T_k(\omega), \infty)}(t), t \in \mathbb{R}.
\]
The filtration generated by process \((N_t)_{t \geq 0}\) is denoted by \((F_t)_{t \geq 0}\). This will be interpreted as the information available to a regular trader at time \(t\).

We define \(S\), a set of simple functions on the Poisson space, by
\[
S := \{ F = f(T_1, \ldots, T_n) : f \in C_c^\infty(\mathbb{R}^n), n \geq 1 \}
\]
and for \(F \in S\) define operator \(D : L^2(B) \to L^2(B \times \mathbb{R}_+)\) by
\[
D_tF = -\sum_{k=1}^{n} 1_{[0,T_k]}(t) \partial_k f(T_1, \ldots, T_n), t \in \mathbb{R}_+.
\]
\(D\) is a closable linear operator (which can be established by using the duality relation, also see Mensi and Privault (2003)). We extend \(S\) into \(\text{Dom} D\) in \(L^2(B)\) with respect to the following norm
\[
||F||_{1,2} := ||F||_{L^2(B)} + ||DF||_{L^2(B \times \mathbb{R}_+)}, F \in S.
\]

We define the space \(D_{1,2}\) as the closure of \(S\) with respect to \(||\cdot||_{1,2}\). It is known (Mensi and Privault (2003)) that \(D\) has an adjoint \(^1\) \(d^* : L^2(B \times \mathbb{R}_+) \to L^2(B)\) which satisfies the duality relation
\[
\mathbb{E} \left[ \int_0^\infty D_tF u_t dt \right] = \mathbb{E}[F d^*(u)], F \in \text{Dom} D, u \in \text{Dom} d^*
\]
and is equivalent to the stochastic integral with respect to the compensated Poisson process for adapted processes in \(\text{Dom} d^*\).

### 3. A Calculus for measure-valued random variables

The basic structure is exactly the same as in Imkeller et al. (2001). Of key importance is the embedding of \(M\), the space of signed measures on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) equipped with variation norm \(\cdot\), into \(\mathbb{R}^N\). We use it to define several important entities; an Itô integral, conditional expectation, etc. The embedding to be used is \(\Phi\),
\[
\Phi : M \to \mathbb{R}^N
\]
\[
\Phi(\mu) = (\langle \mu, f_i \rangle)_{i \in \mathbb{N}}
\]
where \(\langle \mu, f \rangle = \int f d\mu\) and \((f_i)_{i \in \mathbb{N}}\) is a (countable) sequence of functions dense in \(C_b(\mathbb{R})\); clearly, \(\Phi\) is injective, and hence a true embedding into \(\mathbb{R}^N\) due to the density of \((f_i)_{i \in \mathbb{N}}\). And hence the inverse of \(\Phi^{-1}\) could be made sense over the range of \(\Phi\); for the rest of this paper, whenever we write \(\Phi^{-1}\), we take this to be the restriction

\(^1\)We denote the adjoint by \(d^*\) rather than by the usual \(\delta\) to avoid confusion later with the Dirac delta.
map over this range of \( \Phi \), making the inverse map well-defined. We define our set of smooth cylinder measure valued functions to be
\[
S(M) = \{ F = g(T_1, \ldots, T_n, x)dx, g \in C_c^\infty(\mathbb{R}^{n+1}), n \in \mathbb{N} \}
\]
and the measure-valued Poisson Malliavin derivative for \( F \in S(M) \) to be
\[
D_s F = - \sum_{k=1}^n \partial_k g(T_1, \ldots, T_n, x)dx \mathbb{1}_{[0, T_k]}(s)
\]
and \( DF \) is an element of \( L^2(B \times \mathbb{R}_+, M) \).

For clarity, in the next section let \( D^R \) be the gradient operator for the real-valued Poisson Malliavin calculus (as constructed in Mensi and Privault (2003)) and let \( D^M \) be the measure-valued version. The norms on the set of smooth cylinder functions for both gradient operators will bear similar notation.

Then
\[
\langle D^M F, f \rangle = -\sum_{k=1}^n \partial_k g(T_1, \ldots, T_n, x)\mathbb{1}_{[0, T_k]}(s) dx = -\sum_{k=1}^n \mathbb{1}_{[0, T_k]}(s) \partial_k g(T_1, \ldots, T_n, x) f(x) dx
\]
Thus
\[
\langle D^M F, f \rangle = D^R (F, f) \quad \text{for } F \in S(M).
\]
Consequently we have
\[
\Phi(D^M F) = (\langle D^M F, f_i \rangle)_{i \in \mathbb{N}} = (\langle D^R (F, f_i) \rangle)_{i \in \mathbb{N}} \quad \text{for } F \in S(M).
\]
Now introduce a norm on \( S(M) \). For \( F \in S(M) \) let
\[
\| F \|_1^M := E[|F|^2]^{1/2} + E[\|D^M F\|_2^2]^{1/2}.
\]
For clarity, we state here the notation \( \|D^M F\|_2 \) is the \( L^2 \)-norm of \( D^M F \), the variation norm of \( D^M F \) and \( \|f\| \) is the sup norm of \( f \). It can be shown that \( \|F\|_1^M < \infty \) for all \( F \in S(M) \). Now define \( \mathbb{D}_{1,2}(M) \) as the closure of \( S(M) \) with respect to \( \| \cdot \|_1^M \). So \( S(M) \) is dense in \( \mathbb{D}_{1,2}(M) \). Proposition 1 extends relation (1) to all \( F \in \mathbb{D}_{1,2}(M) \).

**Proposition 1.** For \( F \in \mathbb{D}_{1,2}(M) \) and \( f \in C_b(\mathbb{R}) \) we have \( \langle F, f \rangle \in \mathbb{D}_{1,2} \) and \( \langle D^M F, f \rangle = D^R (F, f) \).

Proof. First we prove a useful inequality. For \( F \in S(M) \), \( \langle F, f \rangle \leq \|f\| \|F\| \) and \( D^R (F, f) \leq \| f \| \| D^M F \|. \) Therefore
\[
E[\langle F, f \rangle^2]^{1/2} + E\int_0^1 (D^R (F, f)^2)^{1/2} ds^{1/2} \leq \|f\| E[|F|^2]^{1/2} + \|f\| E\int_0^1 |D^M F|^2 ds^{1/2}
\]
which implies \( \|\langle F, f \rangle\|^R_{1,2} \leq \|f\| \|\|F\|_1^M \).
Now take $F \in \mathbb{D}_{1,2}(\mathbb{M})$. Then there exists a sequence in $S(\mathbb{M})$, $F_n$, such that $F_n \to F$ with respect to $|| \cdot ||_{1,2}$ and so
\[
||\langle F_n, f \rangle - \langle F, f \rangle ||_{1,2} \leq ||f|| \cdot ||F_n - F||_{1,2}
\]
and so if $F_n \to F$ in $\mathbb{D}_{1,2}(\mathbb{M})$ then $\langle F_n, f \rangle \to \langle F, f \rangle$ in $\mathbb{D}_{1,2}$. Since $D^R$ and $D^M$ are closable, $D^R(F_n, f) \to D^R(F, f)$ in $L^2$ and $D^M F_n \to D^M F$ and if $F_n, F \in \mathbb{D}_{1,2}(\mathbb{M})$ we have $||D^M F_n - D^M F, f||_2 \leq ||f|| \cdot ||F_n - F||_{1,2}$ from Equation 3. Then this implies $\langle D^M F_n, f \rangle \to \langle D^M F, f \rangle$ in $L^2$ sense.

Since $D^R(F_n, f) \to D^R(F, f)$ in $L^2$ sense and $D^R(F_n, f) = D^R(F, f)$ for $F \in S(\mathbb{M})$ (see Equation 1), by uniqueness of limits we must have $D^R(F, f) = \langle D^M F, f \rangle$ for $F \in \mathbb{D}_{1,2}(\mathbb{M})$. □

The following proposition extends relation (2) to all $F \in \mathbb{D}_{1,2}(\mathbb{M})$.

**Proposition 2.** For $F \in \mathbb{D}_{1,2}(\mathbb{M})$, $\Phi(D^M F) = ((D^R(F, f_i))_{i \in \mathbb{N}}).

**Proof.** In accordance with the Proposition 1, we have, by definition,
$\Phi(D^M F) = ((D^M F, f_i))_{i \in \mathbb{N}} = (D^R(F, f_i))_{i \in \mathbb{N}}$. □

Our next step is to construct a stochastic integral. Let $(F_t)_{t \in \mathbb{R}_+}$ be a measure-valued process adapted to the natural filtration $\mathcal{F}_t$. By definition, $\Phi(F_t) = ((F_t, f_i))_{i \in \mathbb{N}}$, to say $F_t$ is adapted is to say that for all $i \in \mathbb{N}$, the real-valued process $(F_t, f_i)_{t \in \mathbb{R}_+}$ is adapted. Since $(f_i)$ is dense in $C_b(\mathbb{R})$, then $F_t$ is adapted if and only if $(F_t, f)$ is adapted for any $f \in C_b(\mathbb{R})$. We also require uniform square-integrability of $F_t$ in the sense that,
\[
\mathbb{E} \left[ \int_0^1 |F_t|^2 dt \right] < \infty \quad \text{for any } T > 0.
\]

Now, for every $f \in C_b(\mathbb{R})$, $F_t$ is adapted and satisfies this boundedness condition (4), and so $\int_0^1 \langle F_t, f \rangle d\tilde{N}_t$ is well defined where $\tilde{N}_t$ is the compensated Poisson process. Before we proceed on, we first define the notion of stochastic integral of measure-valued processes against the compensated Poisson process.

**Proposition 3.** For $F_t$ suitably adapted and satisfying (4). There is a unique measure-valued adapted process, $m$, such that for any $f \in C_b(\mathbb{R})$,
\[
\langle m, f \rangle = \int_0^1 \langle F_t, f \rangle d\tilde{N}_t.
\]

From now on, we denote $m$ by $\int_0^\cdot F_t d\tilde{N}_t$.

**Proof.** Note that under Condition 4, by using Doob’s $L^p$-inequality, we can also conclude that
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq 1} \sup_{f \in C_b(\mathbb{R}), ||f|| \leq 1} \left( \int_0^s \langle F_t, f \rangle d\tilde{N}_t \right)^2 \right] 
\leq \mathbb{E} \left[ \sup_{0 \leq s \leq 1} \sup_{f \in C_b(\mathbb{R}), ||f|| \leq 1} \left( \int_0^s |\langle F_t, f \rangle| (dN_t + \lambda dt) \right)^2 \right] 
\leq \mathbb{E} \left[ \sup_{0 \leq s \leq 1} \left( \int_0^s |F_t| (dN_t + \lambda dt) \right)^2 \right]
\]
Therefore, based on the chosen dense sequence \((f_i)_{i \in \mathbb{N}}\), there is a full measure set \(\Omega\), so that for each \(\omega \in \Omega\), there is some \(C(\omega) > 0\),

\[
\left| \int_0^s \langle F_t(\omega), f_i \rangle d\tilde{N}_t(\omega) \right| \leq C(\omega) |f_i|, \quad \text{for any } 0 < s < T,
\]

and it is also a linear functional over the set \(S_\Omega\) of all linear combinations of finitely many \(f_i\) with rational coefficients; and hence this functional could be extended to \(C_b(\mathbb{R})\), and by Riesz representation theorem, there is a finite measure-valued process \(m_s\), such that for any \(i \in \mathbb{N}\) and \(s > 0\),

\[
< m_s(\omega), f_i > = \int_0^s \langle F_t(\omega), f_i \rangle d\tilde{N}_t(\omega).
\]

Now let \((g_j)_j\) be a subsequence of algebraic sums involving only finitely many functions in \((f_i)_i\) such that \(g_j \to f\) in \(C_b(\mathbb{R})\). By using Doob's \(L^p\)-inequality, we have

\[
\mathbb{E} \left[ \sup_{s \in [0,1]} \left( \int_0^s \langle F_t, f - g_j \rangle d\tilde{N}_t \right)^2 \right]^{1/2} \leq 2\mathbb{E} \left[ \int_0^1 \langle F_t, f - g_i \rangle^2 dt \right]^{1/2} \leq 2\mathbb{E} \left[ \int_0^1 |F_t|^2 dt \right]^{1/2} \cdot \|f - g_j\|,
\]

which converges to zero as \(j \to \infty\) under Condition 4, and therefore there is a subsequence \((g_{jk})_k\) so that almost surely, for any \(s > 0\),

\[
\langle m_s, g_{jk} \rangle = \int_0^s \langle F_t, g_{jk} \rangle d\tilde{N}_t \to \int_0^s \langle F_t, f \rangle d\tilde{N}_t.
\]

On the other hand, by the definition, \(m_s\) is a bounded linear operator almost surely, and hence \(\langle m_s, g_{jk} \rangle \to \langle m_s, f \rangle\). Thus, for all \(f \in C_b(\mathbb{R})\) we have

\[
\langle m_s, f \rangle = \int_0^s \langle F_t, f \rangle d\tilde{N}_t,
\]

which is measurable and adapted by the definition. And as a consequence, \(m_s\) can be standardized to become a properly measurable measure-valued adapted process; indeed, this is due to the fact that the Borel sets of \(m_s\) is generated by \(\langle m_s, f \rangle\) for any \(f \in C_b(\mathbb{R})\). The uniqueness property follows from the density of \((f_i)_i\). \(\square\)

Our next aim is to define a conditional expectation formula. Let \(F\) be a \(\mathcal{M}\)-valued random variable that is \(\mathcal{F}\)-measurable such that \(\mathbb{E}(|F|) < \infty\). Then \(\langle F, f_i \rangle\) is \(\mathcal{F}\)-measurable for \(i \in \mathbb{N}\), as is \(\langle F, f \rangle\) for any \(f \in C_b(\mathbb{R})\). Let \(\mathcal{G}\) be a sub-\(\sigma\)-algebra of \(\mathcal{F}\). Then for each \(i\), \(\mathbb{E}(\langle F, f_i \rangle | \mathcal{G})\) is well-defined, up to \(P\)-almost surely equality. Similar
to the proof of Proposition 3, there is a full measure set Ω on which \( \mathbb{E}(\langle F, \cdot \rangle | \mathcal{G}) \) is linear on \( \mathcal{S}_Q \). Note that
\[
\sup_{\|f\| \leq 1} |\mathbb{E}(\langle F, f \rangle | \mathcal{G})| \leq \mathbb{E} \left( \sup_{\|f\| \leq 1} \langle F, f \rangle \right) = \mathbb{E}(\|F\| | \mathcal{G}).
\]

And hence \( \mathbb{E} \left( \sup_{\|f\| \leq 1} |\mathbb{E}(\langle F, f \rangle | \mathcal{G})| \right) \leq \mathbb{E}(\|F\|) < \infty \). Therefore, for every \( \omega \in \Omega \), \( \mathbb{E}(\langle F, \cdot \rangle | \mathcal{G}) \) can be extended to a linear operator on \( C_b(\mathbb{R}) \); an application of Riesz representation theorem, one can guarantee the existence of a measure \( \eta_{\mathcal{G}}(\omega) \), so that for all \( g \in \mathcal{S}_Q \),
\[
\langle \eta_{\mathcal{G}}, g \rangle = \mathbb{E}(\langle F, g \rangle | \mathcal{G}).
\]

Finally, applying the same arguments as in the proof of Proposition 3, e.g. utilizing Doob’s \( L^p \)-inequality, we deduce the measurability of \( \eta_{\mathcal{G}} \). From now on, we denote this \( \eta_{\mathcal{G}} \) by \( \mathbb{E}[F|\mathcal{G}] \), i.e.
\[
\Phi(\mathbb{E}[F|\mathcal{G}]) = (\mathbb{E}(\langle F, f_i \rangle | \mathcal{G})_{i \in \mathbb{N}}.
\]

From the above explanation and the notation adopted, we also have, almost surely,
\[
(5) \quad \mathbb{E}(\langle F, f \rangle | \mathcal{G}) = \langle \mathbb{E}[F|\mathcal{G}], f \rangle \text{ for all } f \in C_b(\mathbb{R}).
\]

Furthermore, the tower property remains valid in this newly defined conditional expectation; indeed, for any \( \mathcal{G} \subseteq \mathcal{H} \), we have
\[
\mathbb{E}\left( \mathbb{E}(F|\mathcal{H}) | \mathcal{G} \right) = \mathbb{E}(F|\mathcal{G}),
\]
which can be proved as follows:
\[
\langle \mathbb{E}(\mathbb{E}(F|\mathcal{H}) | \mathcal{G}), f \rangle = \mathbb{E}(\langle\mathbb{E}(F|\mathcal{H}) | \mathcal{G}, f \rangle \rangle = \mathbb{E}(\langle F, f \rangle | \mathcal{G}) = \langle \mathbb{E}(F|\mathcal{G}), f \rangle,
\]
for any \( f \in C_b(\mathbb{R}) \), where the second last equality follows by the tower property of usual conditional expectations.

We can now prove a measure-valued Clark-Ocone formula. We refer to the classical Clark-Ocone formula in a Poisson setting as proved in Proposition 2 of Mensi and Privault (2003) and Theorem 1 of Privault (1994): For \( F \in \mathcal{L}^2(B) \),
\[
F = \mathbb{E}[F] + \int_0^1 \mathbb{E}[D_t^F F|\mathcal{F}_t] d\tilde{N}_t
\]
We have shown that for \( F \in \mathcal{D}_{1,2}(\mathbb{M}) \), \( \langle F, f \rangle \in \mathcal{D}_{1,2} \). Thus from the classical formula,
\[
\langle F, f_i \rangle = \mathbb{E}[\langle F, f_i \rangle] + \int_0^1 \mathbb{E}[D_t^F \langle F, f_i \rangle | \mathcal{F}_t] d\tilde{N}_t
\]
for all \( i \in \mathbb{N} \). Applying \( \Phi^{-1} \) to both sides, we first observe that the left hand side is equal to \( \Phi^{-1}(\langle F, f_i \rangle) = F \). Since \( \mathbb{E}[\langle F, f_i \rangle] = \mathbb{E}[\langle F, f_i \rangle | \mathcal{F}_0] \), we have \( \Phi^{-1}(\langle \mathbb{E}[\langle F, f_i \rangle] | i \in \mathbb{N} \rangle) = \mathbb{E}[F|\mathcal{F}_0] = \mathbb{E}[F] \) because \( \mathbb{E}(\|F\|) \leq \mathbb{E}(\|F\|^2)^{1/2} < \infty \). Finally,
\[
\mathbb{E}[D_t^F \langle F, f_i \rangle | \mathcal{F}_t] = \mathbb{E}[\langle D_t^M F, f_i \rangle | \mathcal{F}_t] = \langle \mathbb{E}[D_t^M F|\mathcal{F}_t], f_i \rangle
\]
by respectively Proposition 1, and the fact that \( F \in \mathcal{D}_{1,2}(\mathbb{M}) \) implies that \( \mathbb{E}(\|D_t^M F\|^2) < \infty \) for almost every \( t \). Applying Fubini’s theorem twice, we first have
\[
\mathbb{E} \left( \int_0^1 |\mathbb{E}(D_t^M F|\mathcal{F}_t)|^2 dt \right) \leq \mathbb{E} \left( \int_0^1 |D_t^M F|^2 dt \right) < \infty,
\]
and hence Condition 4 is satisfied by $\mathbb{E}(D^M F | \mathcal{F}_t)$, therefore,
\[
\int_0^1 \mathbb{E}[D^R_s F, f_i | \mathcal{F}_s] d\tilde{N}_s = \int_0^1 \langle \mathbb{E}[D^M_t F | \mathcal{F}_t], f_i \rangle d\tilde{N}_t = \langle \int_0^1 \mathbb{E}[D^M t F | \mathcal{F}_t] d\tilde{N}_t, f_i \rangle
\]
from Proposition 3 and so for each $i \in \mathbb{N}$
\[
\langle F, f_i \rangle = \langle \mathbb{E}[F] + \int_0^1 \mathbb{E}[D^M t F | \mathcal{F}_t] d\tilde{N}_t, f_i \rangle.
\]
Applying $\Phi^{-1}$ to both sides once again, we have that for any $F \in D_{1,2}(M)$,
\[
F = \mathbb{E}[F] + \int_0^1 \mathbb{E}[D^M t F | \mathcal{F}_t] d\tilde{N}_t.
\]
Thus we have proven a Clark-Ocone type formula on our measure-valued space and our first theorem.

**Theorem 1.** For every $F \in D_{1,2}(M)$, $F$ has an integral representation given by
\[
F = \mathbb{E}[F] + \int_0^1 \mathbb{E}[D^M t F | \mathcal{F}_t] d\tilde{N}_t.
\]

4. **Stochastic Integral Representations of Conditional Laws and Sufficient Conditions for the Existence of an Information Drift**

We begin this section by rewriting Proposition 3 of Mensi and Privault (2003) in our notation.

**Proposition 4.** Given a random variable $L$, let $P_t(\cdot, dx)$ denote a version of the conditional law of $L$ given $\mathcal{F}$ and assume it has representation
\[
P_t(\cdot, dx) = P_0(\cdot, dx) + \int_0^t \phi_s(\cdot, dx) d\tilde{N}_s
\]
and there exists a $g : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ measurable and a stopping time $S$, such that
\[
1_{\{s \leq S\}} \phi_s(\cdot, dx) = 1_{\{s \leq S\}} g_s(\cdot, x) P_s(\cdot, dx)
\]
then
\[
\tilde{N}_t - \int_0^t g_s(\cdot, L) ds
\]
is a $\mathcal{G}_t$-local martingale up to the time $S$, where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L), t \in \mathbb{R}^+$.

Thus Proposition 4 provides sufficient conditions on the conditional law of $L$ for the existence of an information drift. It assumes the existence of an integral representation with a suitable measure-valued random process $\phi_s(\cdot, dx)$ and the absolute continuity of this measure with respect to the original conditional law $P_t(\cdot, dx)$ for each time $s$. However, $\phi_s(\cdot, dx)$ is not given. Fortunately, Theorem 1 tells us that if $P_t(\cdot, dx) \in D_{1,2}(M)$ for each $t$, then $P_t(\cdot, dx)$ has integral representation via
\[
P_t(\cdot, dx) = \mathbb{E}[P_t(\cdot, dx)] + \int_0^1 \mathbb{E}[D^M s P_t(\cdot, dx) | \mathcal{F}_s] d\tilde{N}_s
\]
and consequently we are very close to identifying the process $\phi_x(\cdot, dx)$ in Proposition 4. Note that $P_t(\cdot, dx)$ may be expressed as $P(L \in dx|\mathcal{F}_t)$, hence
\[
\mathbb{E}[P(L \in dx|\mathcal{F}_t)] = \mathbb{E}[\mathbb{E}[\mathbb{E}[1_{L \in dx}|\mathcal{F}_t]|\mathcal{F}_0] = \mathbb{E}[\mathbb{E}[1_{L \in dx}|\mathcal{F}_0] = P_0(\cdot, dx).
\]
To have the stochastic integral in the correct form, it will be sufficient to show that $D_s^B P_t(\cdot, dx) = 0$ when $s \geq t$. This would be another example of a familiar property in Malliavin calculus: the Malliavin gradient of an $\mathcal{F}_t$-adapted process vanishes if the time parameter of the gradient is greater than $t$. Corollary 5.7 in Øksendal (1997) proves the Wiener space case and Proposition 9.3 in Privault (1994) does the same on the Poisson space. We prove our measure-valued, Poisson space based version here. We shall prove the result in the ordinary Poisson Malliavin calculus and then carry it over to the measure-valued version using propositions in the previous section.

Space $\mathcal{V}$ is defined as the class of processes $v \in \mathcal{L}^2(B \times \mathbb{R}_+)$ of the form
\[
v(t) = f(t, T_1, \ldots, T_n)
\]
where $f \in C^\infty(\mathbb{R}^{n+1})$, $n \in \mathbb{N}$ and $f(y, x_1, \ldots, x_n) = 0$ if $y > x_n$. Remark 2 of Privault (1994) tells us that $\mathcal{V}$ is dense in $\mathcal{L}^2(B \times \mathbb{R}_+)$.\[\text{Proposition 5. Let } v \text{ be a predictable process on the filtered Poisson space } (B, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P). \text{ Then we have}
\]
\[
D_s^B v(t) = 0 \text{ a.s. when } s \geq t.
\]
\[\text{Proof. We first note that for any } \mathcal{F}_t\text{-measurable random variable } f, \text{ we have}
\]
\[
f = \mathbb{E}(f|\mathcal{F}_t) = g((N_s)_{s \leq t}) = \sum_{k=0}^{\infty} g((N_s)_{s \leq t}) \mathbb{1}_{\{N_s = k\}}
\]
\[
= \sum_{k=0}^{\infty} g_k(T_1, T_2, \ldots, T_k) \mathbb{1}_{\{T_1 \leq t, i \leq k\}} \mathbb{1}_{\{T_{k+1} > t\}},
\]
for some measurable functions $g$ and $g_k$. By using common density argument, the last functional can be well-approximated by linear combinations of the functions of the form:
\[
V := h(T_1, \ldots, T_k) \mathbb{1}_{\{T_1 \leq t, i \leq k\}} \mathbb{1}_{\{T_{k+1} > t\}},
\]
where $h$ is a smooth function, and we also have for any $s \geq t$,
\[
D_s^B V = \left( \sum_{j=1}^{k} \partial_j h \mathbb{1}_{\{0, T_j\}}(s) \right) \mathbb{1}_{\{T_1 \leq t, i \leq k\}} \mathbb{1}_{\{T_{k+1} > t\}} + h \mathbb{1}_{\{T_1 \leq t, i \leq k\}} (\partial_{k+1} \mathbb{1}_{\{T_{k+1} > t\}})
\]
\[
= \left( \sum_{j=1}^{k} \partial_j h \delta_{\{T_j = s\}} \right) \mathbb{1}_{\{T_1 \leq t, i \leq k\}} \mathbb{1}_{\{T_{k+1} > t\}} + h \mathbb{1}_{\{T_1 \leq t, i \leq k\}} (\delta_{\{T_{k+1} = s\}})
\]
\[= 0 \text{ a.s.,}
\]
here $\delta_{\{T_j = s\}}$ is the Dirac delta function, and the last equality holds since $\mathbb{P}(T_j = s) = 0$ for any fixed $s$, $j = 1, \ldots, k + 1$, as $N$ is a Poisson process. And hence the same result could be deduced for any $\mathcal{F}_t$-measurable random variable $f \in \mathbb{D}_{1,2}$.

Now, for a process $v$ of the form $v(u) = f(T_1, \ldots, T_n) \mathbb{1}_{\{t, \infty\}}(u)$ for some $t > 0$ with $f \in C^\infty(\mathbb{R}_n)$, $f(T_1, \ldots, T_n)$ and $\mathcal{F}_t$-measurable, is called a cylindrical elementary predictable process. Based on the previous arguments, $D_s^B v(u) =$
\( \mathbb{I}_{[t,\infty)}(u)D^R_s f(T_1, \ldots, T_n) \) which vanishes for \( s \geq t \). Because \( D^R \) is a linear operator, this result also holds for processes \( \psi \) that are linear combinations of cylindrical elementary predictable processes. A familiar density argument extends the result to all predictable processes in \( L^2(\mathcal{B} \times \mathbb{R}_+) \).

**Corollary 1.** Let \( u(t, \cdot) \) be a measure-valued \( \mathcal{F}_t \)-adapted stochastic process s.t. \( u(t, \cdot) \in \mathcal{D}_{1,2}(\mathcal{M}) \) for all \( t \in [0, 1] \). Then \( D^M_s u_t = 0 \) a.s. when \( s \geq t \).

**Proof.** As remarked upon before when considering the measure-valued stochastic integral, to say that a measure-valued random process \( u_t \) is \( \mathcal{F}_t \)-adapted is to say that \( \langle u_t, f \rangle \) is adapted for any \( f \in C_b(\mathbb{R}) \). By Proposition 1 if \( u_t \in \mathcal{D}_{1,2}(\mathcal{M}) \) for all \( t \in [0, 1] \), then \( \langle u_t, f \rangle \in \mathcal{D}_{1,2} \) for each \( t \in [0, 1] \). Then consider \( D^R_s \langle u_t, f \rangle \). Since \( \langle (u_t, f) \rangle \) is an \( \mathcal{F}_t \)-adapted process, by Proposition 5 \( D^M_s \langle u_t, f \rangle = 0 \) when \( s \geq t \). Thus by Proposition 1 \( D^M_s u_t = D^R_s u_t = 0 \) when \( s \geq t \). Since this is true for all \( f \in C_b(\mathbb{R}) \), \( D^M_s u_t \) must equal zero when \( s \geq t \). \( \square \)

Since \( P_t(\cdot, dx) \) is a \( \mathcal{F}_t \)-adapted measure-valued process, Corollary 1 applies and we have \( D^M_s P_t(\cdot, dx) = 0 \) when \( s \geq t \). We have proved the following proposition, as an extension to Proposition 3 in Mensi and Privault (2004).

**Proposition 6.** Given a random variable \( L \) and \( P_t(\cdot, dx) \) its conditional law given \( \mathcal{F}_t \), then if \( P_t(\cdot, dx) \in \mathcal{D}_{1,2}(\mathcal{M}) \) for each \( t \), then \( P_t(\cdot, dx) \) has integral representation via

\[
P_t(\cdot, dx) = P_0(\cdot, dx) + \int_0^t \mathbb{E}[D^M_s P_t(\cdot, dx) | \mathcal{F}_s] d\mathbb{N}_s
\]

and if there exists a \( g : \mathbb{R}^+ \times B \times \mathbb{R}^+ \to \mathbb{R} \) measurable and stopping time \( S \) such that

\[
\mathbb{I}_{\{s \leq S\}} \mathbb{E}[D^M_s P_t(\cdot, dx) | \mathcal{F}_s] = \mathbb{I}_{\{s \leq S\}} g_s(\cdot, x) P_s(\cdot, dx)
\]

then

\[
\hat{\mathbb{N}}_t = \int_0^t g_s(\cdot, L) ds
\]

is a \( \mathcal{G}_t \)-local martingale up to the time \( S \), where \( \mathcal{G}_t = \mathcal{F}_t \cap \sigma(L), t \in \mathbb{R}^+ \).

Thus Proposition 6 identifies the process \( \phi_s(\cdot, dx) \) upon whose behaviour the existence of an information drift \( g_s(\cdot, L) \) depends. We would like to find another, simpler form for \( \phi_s(\cdot, dx) \), one that allows us to take advantage of the numerous gradient operators on the Poisson space and provides an elegant extension to Jacod’s condition for the existence of an information drift. Again, Imkeller et al. (2001) leads the way and the similarities between the Poisson and Wiener spaces allows for an easy transition of their results. Imkeller et al. (2001) perturb the random variable \( L \) by a small amount, find an integral representation for the conditional law of this perturbed random variable \( L' \) and look at this representation in the limit as \( \epsilon \) tends to zero. The result is a new expression for \( \phi_s(\cdot, dx) \). We follow their method below and again it should be noted that due to the similarities of the two Malliavin calculi, our calculations rarely differ from those in Imkeller et al. (2001).

We state Theorem 2 here and devote the rest of this section to its proof. For \( \epsilon > 0 \), we define \( p_\epsilon \) as the probability density function of \( \sqrt{\epsilon} Z \), where \( Z \) is a standardized normal random variable independent of \( \mathcal{F}_1 \).
Then we have

\[ E[\frac{L-x}{\epsilon}p_\epsilon(x-L)D_s^R L|\mathcal{F}_s]dx. \]

Then if there exists a measure-valued process \( k_s(\cdot, dx) \) such that for any \( f \in C_b(\mathbb{R}) \), for any \( t \in [0,1] \),

\[ \mathbb{E}\left[ \int_0^t (k^\epsilon_s(\cdot, dx) - k_s(\cdot, dx), f)^2 ds \right] \to 0 \quad \text{as} \quad \epsilon \to 0 \]

and

\[ \mathbb{E}\left[ \int_0^t |k_s(\cdot, dx)|^2 ds \right] < \infty, \]

we have the following integral representation for the conditional law of \( L \)

\[ P_t(\cdot, dx) = P_0(\cdot, dx) + \int_0^t k_s(\cdot, dx)d\hat{N}_s \]

and consequently if there exists a \( g : \mathbb{R}_+ \times B \times \mathbb{R}_+ \rightarrow \mathbb{R} \) measurable and stopping time \( S \) such that

\[ 1_{\{s \leq S\}}k_s(\cdot, dx) = 1_{\{s \leq S\}}g_s(\cdot, x)P_s(\cdot, dx) \]

then

\[ \hat{N}_t - \int_0^t g_s(\cdot, L)ds \]

is a \( \mathcal{G}_t \)-local martingale up to the time \( S \), where \( \mathcal{G}_t = \mathcal{F}_t \vee \sigma(L) \), \( t \in \mathbb{R}_+ \).

**Proof.** Let \( L \in \mathbb{D}_{1,2} \) and \( Z \) be a \( N(0,1) \)-variable on \( (B, \mathcal{F}, P) \) which is independent of \( \mathcal{F}_1 \). For \( \epsilon > 0 \), let \( L_\epsilon = L + \sqrt{\epsilon}Z \) and \( P^\epsilon_t(\cdot, dx) \) be a version of the regular conditional law of \( L_\epsilon \) given \( \mathcal{F}_t \), \( 0 \leq t \leq 1 \), that is \( P^\epsilon_t(\cdot, dx) = P(L_\epsilon \in dx|\mathcal{F}_t) \) for \( 0 \leq t \leq 1 \).

Assume that \( P^\epsilon_t(\cdot, dx) \in \mathbb{D}_{1,2}(\mathbb{M}) \). Proposition 1 tells us that \( \langle P^\epsilon_t(\cdot, dx), f \rangle \in \mathbb{D}_{1,2} \) for \( f \in C_b(\mathbb{R}) \). Then by using the classical Clark formula in Proposition 2 of Mensi and Privault (2004) as well as recalling Proposition 5 we have

\[ \langle P^\epsilon_t(\cdot, dx), f \rangle = \mathbb{E}[\langle P^\epsilon_t(\cdot, dx), f \rangle] + \int_0^t \mathbb{E}[D^R_s \langle P^\epsilon_t(\cdot, dx), f \rangle|\mathcal{F}_s]d\hat{N}_s. \]

Consider \( \mathbb{E}[D^R_s \langle P^\epsilon_t(\cdot, dx), f \rangle|\mathcal{F}_s] \). We have \( \langle P^\epsilon_t(\cdot, dx), f \rangle = \mathbb{E}[f(L_\epsilon)|\mathcal{F}_t] \) by definition. Then we have

\[ \langle P^\epsilon_t(\cdot, dx), f \rangle = \mathbb{E}[\mathbb{E}[f(L_\epsilon)|\mathcal{F}_s]|\mathcal{F}_t] \]

\[ = \mathbb{E}\left[ \int_R f(L + x)p_\epsilon(x)dx|\mathcal{F}_t \right] \]

\[ = \mathbb{E}\left[ \int_R f(y)p_\epsilon(y-L)dy|\mathcal{F}_t \right] \]

\[ = \int_R \mathbb{E}[p_\epsilon(y-L)|\mathcal{F}_t]f(y)dy \]

\[ = \langle (\mathbb{E}[p_\epsilon(y-L)|\mathcal{F}_t]f(y), f \rangle \]

Hence \( D^R_s \langle P^\epsilon_t(\cdot, dx), f \rangle = D^R_s \langle \mathbb{E}[p_\epsilon(y-L)|\mathcal{F}_t]f(y), f \rangle = D^R_s \langle \mathbb{E}[p_\epsilon(y-L)|\mathcal{F}_t]f(y), f \rangle \) for \( s \leq t \), following Proposition 1. The measure-valued random variable \( \mathbb{E}[p_\epsilon(y-L)|\mathcal{F}_t]f(y) \) is continuous with respect to the Lebesgue measure. This fact and the given similarities in the definitions of operators \( D^M \) and \( D^R \) allow us to write
provided that and \( \| \cdot \| \) interchange

\[
\int_0^1 D_s^R(E[F|\mathcal{F}_t]) \mathbb{1}_{[u,v] \times A_u}(s, \omega) ds = E[\mathbb{E}[F|\mathcal{F}_t]d^*(\mathbb{1}_{[u,v] \times A_u}(s, \omega))].
\]

Since \( \mathbb{1}_{[u,v] \times A_u}(s, \omega) \) is a \( \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t \)-adapted process, the adjoint \( d^*(\cdot) \) is equal to the stochastic integral, therefore

\[
E[\int_0^1 D_s^R(E[F|\mathcal{F}_t]) \mathbb{1}_{[u,v] \times A_u}(s) ds] = E[\mathbb{E}[F|\mathcal{F}_t] \int_0^1 \mathbb{1}_{[u,v] \times A_u}(s)d\hat{N}_s]
\]

\[
= E\left[\mathbb{E}[F|\mathcal{F}_t] \mathbb{1}_{A_u}(s)(\hat{N}_v - \hat{N}_u)\right]
\]

\[
= E\left[\mathbb{E}[F]\mathbb{1}_{A_u}(\hat{N}_v - \hat{N}_u)|\mathcal{F}_t]\right]
\]

\[
= E\left[F\int_0^1 \mathbb{1}_{[u,v] \times A_u}(s)d\hat{N}_s\right]
\]

\[
= E\left[\int_0^1 D_s^R F \mathbb{1}_{[u,v] \times A_u}(s)ds|\mathcal{F}_t\right]
\]

\[
= E\left[\int_0^1 E[D_s^R F|\mathcal{F}_t] \mathbb{1}_{[u,v] \times A_u}(s) ds\right]
\]

Since \( \mathbb{1}_{[u,v] \times A_u}(s) \) is a building block for any process adapted to \( \mathcal{F}_t \), the relation holds for all adapted processes and hence we can interchange \( D_s^R \) and \( E \). Consequently, because \( D_s^R \) follows the chain rule, \( D_s^R p_r(y - L) = \frac{-(y-L)}{\epsilon} p_r(y - L) D_s^R L \) and \( \| \frac{y-L}{\epsilon} p_r(y - \cdot) \|_{\infty} < \infty \), and so \( p_r(y - L) \in \mathbb{D}_{1,2} \),

\[
E[D_s^R(p_r(y - L))|\mathcal{F}_t]dy = E[D_s^R L \frac{L-y}{\epsilon} p_r(y - L)|\mathcal{F}_t]dy
\]

which implies, for each \( s \),

\[
E[D_s^R(P_s^{*}(\cdot, dx), f)|\mathcal{F}_s] = E[\langle E[D_s^R L \frac{L-y}{\epsilon} p_r(y - L)|\mathcal{F}_t]dy, f\rangle|\mathcal{F}_s]
\]

\[
= \langle E[\frac{L-y}{\epsilon} p_r(y - L) D_s^R L|\mathcal{F}_s]dy, f\rangle
\]

\[
= \langle E[\frac{L-y}{\epsilon} p_r(y - L) D_s^R L|\mathcal{F}_s]dy, f\rangle,
\]

provided that

\[
E\left(\left| \frac{L-y}{\epsilon} p_r(y - L) D_s^R L \right| dy\right) < \infty.
\]

Then we have

\[
\int_0^t E[D_s^R(P_s^{*}(\cdot, dx), f)|\mathcal{F}_s]d\hat{N}_s = \int_0^t \langle E[\frac{L-y}{\epsilon} p_r(y - L) D_s^R L|\mathcal{F}_s]dy, f\rangle d\hat{N}_s
\]

\[
= \langle \int_0^t E[\frac{L-y}{\epsilon} p_r(y - L) D_s^R L|\mathcal{F}_s]dy, f\rangle
\]
provided that
\begin{equation}
E \left( \int_0^1 \left\| \left( \frac{L - y}{\epsilon} p_\epsilon(y - L) D^R_s L \right) dy \right\|^2 ds \right) < \infty.
\end{equation}

Thus we have shown that
\begin{equation}
\langle P_\epsilon^t(\cdot, dx), f \rangle = \langle P_0^t(\cdot, dx), f \rangle + \left( \int_0^t (E_0 \left[ \frac{L - y}{\epsilon} p_\epsilon(y - L) D^R_s L | \mathcal{F}_s \right] dy) d\tilde{N}_s, f \rangle
\end{equation}
for all \( f \in C_b(\mathbb{R}) \). So we must have
\begin{equation}
\left( \int_0^t (E_0 \left[ \frac{L - y}{\epsilon} p_\epsilon(y - L) D^R_s L | \mathcal{F}_s \right] dy) d\tilde{N}_s, f \rangle = \left( \int_0^t E[D^M_s P_\epsilon^t(\cdot, dx) | \mathcal{F}_s] d\tilde{N}_s, f \rangle
\end{equation}
for all \( f \in C_b(\mathbb{R}) \). This implies
\begin{equation}
\int_0^t (E_0 \left[ \frac{L - y}{\epsilon} p_\epsilon(y - L) D^R_s L | \mathcal{F}_s \right] dy) d\tilde{N}_s = \int_0^t E[D^M_s P_\epsilon^t(\cdot, dx) | \mathcal{F}_s] d\tilde{N}_s.
\end{equation}
We can use this representation in the measure-valued Clark-Ocone formula for \( P_\epsilon^t(\cdot, dx) \). Recall we needed to verify the two boundedness Conditions 6 and 7 for this representation to be valid, allowing us to interchange \( \langle \cdot, \cdot \rangle \) with conditional expectations and stochastic integrals. Firstly, note that by using Fubini’s theorem, Condition 7 guarantees that
\begin{equation}
E \left( \left\| \left( \frac{L - y}{\epsilon} p_\epsilon(y - L) D^R_s L \right) dy \right\|^2 \right) \leq E \left( \left\| \left( \frac{L - y}{\epsilon} p_\epsilon(y - L) D^R_s L \right) dy \right\|^2 \right) < \infty,
\end{equation}
almost every \( s \in [0, 1] \), and so Condition 6 holds on a dense set of \( s \) which suffices for our proof. Secondly, for each \( \epsilon > 0 \), we have:
\begin{equation}
E \left( \int_0^1 \left\| \left( \frac{L - y}{\epsilon} p_\epsilon(y - L) D^R_s L \right) dy \right\|^2 ds \right) \leq E \left( \int_0^1 \left( \frac{1}{\epsilon} E_Z(\|Z\| | D^R_s L) \right)^2 ds \right)
= \frac{1}{\epsilon^2} (E_Z(\|Z\|))^2 E \left( \int_0^1 |D^R_s L|^2 ds \right)
\leq \frac{1}{\epsilon^2} \cdot \epsilon \cdot (\|L\|_{1,2}^3)^2 < \infty,
\end{equation}
and hence Condition 7 is satisfied.

Finally, we need to show that we may apply the measure-valued Clark-Ocone formula to \( P_\epsilon^t(\cdot, dx) \) in the first place. That is, we need to show \( P_\epsilon^t(\cdot, dx) \in \mathbb{D}_{1,2}(\mathcal{M}) \) which requires showing \( E[|P_\epsilon^t(\cdot, dx)|^2] < \infty \) and \( E[\int_0^1 |D^M_s P_\epsilon^t(\cdot, dx)|^2 ds] < \infty \). On the one hand,
\begin{equation}
E[|P_\epsilon^t(\cdot, dx)|^2] = E[(\sup_{f \in C_b(\mathbb{R}), |f| \leq 1} \langle P_\epsilon^t(\cdot, dx), f \rangle)^2],
\end{equation}
since as a contraction, \( \langle P_\epsilon^t(\cdot, dx), f \rangle = E[f(L^\epsilon) | \mathcal{F}_t] \leq |f| \) we have \( E[|P_\epsilon^t(\cdot, dx)|^2] < \infty \). The other criterion for \( D^M_s P_\epsilon^t(\cdot, dx) \) is satisfied by a similar proof to above.
\begin{equation}
P_\epsilon^t(\cdot, dx) = P_0^t(\cdot, dx) + \int_0^t k^\epsilon_s(\cdot, dx) d\tilde{N}_s.
\end{equation}

Having perturbed \( L \) by a small amount and derived a new integral representation for \( L^\epsilon \), our next goal is to take the limit as \( \epsilon \to 0 \) and see what happens to the
term inside the stochastic integral. Again, we start in the classical framework and for $f \in C_b(\mathbb{R})$

$$\langle P^v_t(\cdot, dx), f \rangle = \langle P^v_0(\cdot, dx), f \rangle + \int_0^t k^v_s(\cdot, dx) d\tilde{N}_s,$$

Now $\langle P^v_t(\cdot, dx), f \rangle = \mathbb{E}[f(L + \sqrt{\epsilon}N)|\mathcal{F}_t]$ and as $\epsilon \to 0$, $f(L + \sqrt{\epsilon}N) \to f(L)$ in probability. Also, $|f(L + \sqrt{\epsilon}N)| \leq K$, for some $K \in \mathbb{R}_+$ because $f$ is continuous and bounded. Hence $f(L + \sqrt{\epsilon}N) \to f(L)$ in $L^1$. Then by Jensen,

$$\mathbb{E}[\mathbb{E}[f(L + \sqrt{\epsilon}N) - f(L)|\mathcal{F}_t]] \leq \mathbb{E}[|f(L + \sqrt{\epsilon}N) - f(L)|] \to 0 \text{ as } \epsilon \to 0.$$ 

Thus $\langle P^v_t(\cdot, dx), f \rangle \to \langle P^v_0(\cdot, dx), f \rangle$ in $L^1$ as $\epsilon \to 0$. We are left to show the stochastic integral converges. Since the integral is defined in terms of $L^2$-convergence we say that if there exists an $M$-valued process $k_s(\cdot, dx)$ such that for any $t \in [0, 1], f \in C_b(\mathbb{R}),$

$$\mathbb{E}\left[\left(\int_0^t (k^v_s(\cdot, dx) - k_s(\cdot, dx), f)^2 ds\right) \to 0 \text{ as } \epsilon \to 0 \right]$$

then

$$\int_0^t (k^v_s(\cdot, dx), f)d\tilde{N}_s \to \int_0^t (k_s(\cdot, dx), f)d\tilde{N}_s \text{ as } \epsilon \to 0.$$ 

In which case we have

$$(8) \quad \langle P^v_t(\cdot, dx), f \rangle = \langle P^v_0(\cdot, dx), f \rangle + \int_0^t (k_s(\cdot, dx), f)d\tilde{N}_s.$$ 

In order to remove the brackets from relation (8), $k_s(\cdot, dx)$ must satisfy boundedness Condition (4) as required in the statement of the present theorem. If it does, we have a new integral representation for the conditional law and an extension to Proposition 3 of Mensi and Privault (2004) quickly follows. \hfill $\square$

5. Examples of enlargements of filtrations

We now employ the results derived in the previous sections to calculate explicit enlargement of filtrations for a few choices of $L$. We have shown that there are at least three methods for finding the integrand $k_s(\cdot, dx)$ in the stochastic integral representation of $P^v_t(\cdot, dx)$, provided the relevant boundedness conditions are met:

- Calculate $\mathbb{E}[D^M_s P^v_t(\cdot, dx)|\mathcal{F}_t]$.
- Calculate $k^v_s(\cdot, dx) = \mathbb{E}[\frac{L-z}{z}p_N(x - L)D^R_s L|\mathcal{F}_s]dx$ and take limits as $\epsilon \to 0$.
- Calculate $k'_s(\cdot, dx) = \mathbb{E}[D^R_s E[p_N(x - L)]|\mathcal{F}_s]dx$ and take limits as $\epsilon \to 0$.

We shall see that the last approach is the easiest in practice as it permits a finite-difference operator to be used in place of $D^R$.

Nualart and Vives (1990) defines a derivative operator $D$ on the Fock space associated to square-integrable functions on a measure space, $L^2(T, \mathcal{B}, \lambda)$. From its definition, $D$ is an annihilation operator. By specifying measure space $(T, \mathcal{B}, \lambda)$ and defining the canonical Poisson space $(\Omega, \mathcal{F}, P)$ over this measure space, the authors show that the Fock space associated to $L^2(T, \mathcal{B}, \lambda)$ is isometric to $L^2(\Omega, \mathcal{F}, P)$. This allows the derivative operator $D$ to be interpreted on the Poisson space and indeed Theorem 6.2 of Nualart and Vives (1990) shows that it is equal to a translation operator $\Psi$ on Poisson functionals in the domain of $D$.

Nualart and Vives (1990) define their translation operator $\Psi_t$ on a Poisson space in the following way. Let $(T, \mathcal{B}, \lambda)$ be a measure space with $T$ locally compact with a countable basis (e.g. $[0,T]$), $\mathcal{B}$ is the Borel $\sigma$-algebra on $T$ (e.g. $\mathcal{B}([0,T])$, and $\lambda$
is a Radon measure that charges all the open sets and is diffuse over the \( \sigma \)-field \( \mathcal{B} \) (e.g. \( \lambda dx, \lambda \) a positive constant). Then define the Poisson space over this measure space by taking

\[
\Omega := \{ \omega = \sum_{j=0}^{n} \delta_{t_j}, n \in \mathbb{N} \cup \{ \infty \}, t_j \in T \}
\]

\[
F_0 := \sigma\{ p_A : p_A(\omega) = \omega(A), A \in \mathcal{B} \}
\]

with probability measure \( \mathbb{P} \) defined over \( (\Omega, \mathcal{F}_0) \) such that

\[
\mathbb{P}(p_A = k) = e^{-\lambda(\omega)} \frac{\lambda(\omega)^k}{k!}, k \geq 0, A \in \mathcal{B}
\]

for all \( A, B \in \mathcal{B}, A \cap B = \emptyset, p_A \) and \( p_B \) are \( \mathbb{P} \)-independent. We complete \( \mathcal{F}_0 \) by \( P \) to \( (\Omega, \mathcal{F}, \mathbb{P}) \) as our Poisson triple. Then the translation operator on \( \Omega \) is defined by

\[
\Psi_t(\omega) = \omega + \delta_t\text{ and } (\Psi_t(F))(\omega) = F(\omega + \delta_t) - F(\omega)
\]

for a Poisson functional \( F \).

Propositions 20 and 21 of Privault (1994) tell us that for \( F \in \mathcal{L}^2(B) \), with \( B \) the sequence of the Poisson space used in this paper and \( \tilde{D} \) a continuous-time derivative, we have

\[
E[\tilde{D}_tF|\mathcal{F}_t] = E[\Psi_tF|\mathcal{F}_t]
\]

and as a result, any \( F \in \mathcal{L}^2(B) \) has the integral representation

\[
F = E[F] + \int_0^1 E[\Psi_tF|\mathcal{F}_t]d\tilde{N}_t.
\]

Thus \( \Psi \) provides an alternative way to calculate \( k^*_s(\cdot, dx) \) and we are now ready to calculate explicit enlargement of filtrations. Our first example is that provided in Mensi and Privault (2003).

5.1. \( L = \tau_{N_T} \). This choice of \( L \) represents the insider knowing the length of the interarrival time straddling \( T \), the final trading time. From now on, \( T \) shall replace 1 as the end of the trading interval and all the results in the previous sections can be modified accordingly. Mensi and Privault (2004) calculate the conditional law of \( \tau_{N_T} \) as

\[
P_t(\cdot, dx) = (1_{[0,\infty)}(x)e^{-x}(T - t) \wedge x + e^{t-T_{N_t}}x^21_{[T-T_{N_t},\infty)}(x))dx
\]

and from this \( \phi_s(\cdot, dx) \) is found using a real valued Clark formula with respect to a finite difference operator \( \nabla \) and the fact that \( P_t(\cdot, dx) \) has a density.

Our calculations begin with the fact that

\[
k^*_s(\cdot, dx) = E[D_sE[p_s(x - L)|\mathcal{F}_t]|\mathcal{F}_t]dx = E[\Psi_sE[p_s(x - L)|\mathcal{F}_t]|\mathcal{F}_s]dx
\]

and using the conditional law of \( L \) as given in (9)

\[
\Psi_sE[p_s(x - L)|\mathcal{F}_t] = \Psi_s(\int_0^{t-N_t} e^{-y}(T - t) \wedge y e^{-\frac{1}{2}(x-y)^2/\sqrt{2\pi t}}dy + e^{t-N_t} \int_{T-N_t}^{\infty} e^{-y} e^{-\frac{1}{2}(x-y)^2/\sqrt{2\pi t}}dy)
\]

\[
= \Psi_s(e^{t-N_t} \int_{T-N_t}^{\infty} e^{-y} e^{-\frac{1}{2\pi}(x-y)^2/\sqrt{2\pi t}}dy)
\]
Now we work out $\Psi_s(T_{N_t})$ as $T_{N_t}$ is the only random variable that appears in $E[p_e(x-L)|F_t]$. In the notation of Nualart and Vives (1990),

$$T_{N_t}(\omega) = \inf_{u \in [0,T]} \{\omega[0,u] = \omega[0,t]\} \text{ and } T_{N_t}(\omega + \delta_s) = \inf_{u \in [0,T]} \{\omega[0,u] + \delta_s[0,u] - \omega[0,t] = \delta_s[0,t]\}$$

Thus if $s > t$, then $\Psi_s(T_{N_t}) = T_{N_t} - T_{N_s}$. If $s < T_{N_t}$, then $\Psi(T_{N_t}) = T_{N_t} - T_{N_s}$. If $T_{N_t} \leq s \leq t$, then $\Psi(T_{N_t}) = s$. Altogether we have, $\Psi_s(T_{N_t}) = (s - T_{N_t})\mathbb{1}_{\{T_{N_t} \leq s \leq t\}}$ and for a functional $f$,

$$\Psi_s f(T_{N_t}) = \mathbb{1}_{\{T_{N_t} \leq s \leq t\}}(f(s) - f(T_{N_t})).$$

In our case, our functional is given by

$$f(\cdot) = e^{t-\cdot} \int_{T-t}^{\infty} e^{-y} e^{-\frac{1}{2}(x-y)^2} \sqrt{2\pi} \, dy,$$

so we have

$$\Psi_s \mathbb{E}[p_e(x-L)|F_t] = \mathbb{1}_{\{T_{N_t} \leq s \leq t\}}(e^{t-s} \int_{T-s}^{\infty} e^{-y} e^{-\frac{1}{2}(x-y)^2} \sqrt{2\pi} \, dy$$

$$- e^{t-T_{N_t}} \int_{T-T_{N_t}}^{\infty} e^{-y} e^{-\frac{1}{2}(x-y)^2} \sqrt{2\pi} \, dy).$$

After some simplifications we see that

$$(10) \quad \mathbb{E}[\Psi_s \mathbb{E}[p_e(x-L)|F_t]|F_s] = \mathbb{1}_{\{s \leq t\}} e^{t-s} \int_{T-s}^{\infty} g(y)dy \mathbb{E}[\mathbb{1}_{\{T_{N_t} \leq s\}}|F_s]$$

$$- \mathbb{1}_{\{s \leq t\}} \mathbb{E}[\mathbb{1}_{\{t-T_{N_t} \geq t-s\}} e^{t-T_{N_t}} \int_{T-t+t-T_{N_t}}^{\infty} g(y)dy|F_s]$$

where $g(y)$ is $e^{-y} e^{-\frac{1}{2}(x-y)^2} \sqrt{2\pi}$. We now use the conditional law of $T_{N_t}$ given $F_s$:

$$P(T_{N_t} > s|F_s) = 1 - \mathbb{1}_{\{s \leq t\}} e^{-(t-s)}$$

and the conditional law of $t-T_{N_t}$ given $F_s$

$$dP(t-T_{N_t} = x|F_s) = \mathbb{1}_{\{0,t-s\}}(x)e^{-x}dx + e^{-(t-s)} \delta_{t-T_{N_t}}(dx).$$

The right hand side of expression (10) is

$$\mathbb{1}_{\{s \leq t\}} \int_{T-s}^{\infty} g(y)dy$$

and expression (11) is equal to

$$\mathbb{1}_{\{s \leq t\}} e^{s-T_{N_s}} \int_{T-T_{N_s}}^{\infty} g(y)dy.$$

We are now in a position to find $k_s(\cdot,dx)$ by taking the limit of the above two parts as $\epsilon$ tends to zero. In doing so we find that

$$\lim_{\epsilon \to 0} \mathbb{1}_{\{s \leq t\}} \int_{T-s}^{\infty} g(y)dy dx = \mathbb{1}_{\{s \leq t\}} \mathbb{1}_{\{x \geq t-s\}} e^{-x}dx$$

$$\lim_{\epsilon \to 0} \mathbb{1}_{\{s \leq t\}} e^{s-T_{N_s}} \int_{T-T_{N_s}}^{\infty} g(y)dy dx = \mathbb{1}_{\{s \leq t\}} e^{s-T_{N_s}} \mathbb{1}_{\{x \geq t-T_{N_s}\}} e^{-x}dx.$$
with limits being taken in an $L^2$ sense. Thus $k_s(\cdot, dx)$ above is exactly the same as the $\phi_s(\cdot, dx)$ ($\lambda_t(dx)$ in their notation) found in Mensi and Privault (2003). After a little rearranging we see that

$$
k_s(\cdot, dx) = \mathbb{I}_{\{T-s \leq x < T-N_n\}} e^{-x} + \mathbb{I}_{\{T-N_n \leq x\}} (e^{-x} + e^{T-N_n-x})dx
$$

$$
\phi_s(\cdot, dx) = \mathbb{I}_{\{0 \leq x < T-s\}} xe^{-x} + \mathbb{I}_{\{T-s \leq x < T-N_n\}} (T-s)e^{-x} + \mathbb{I}_{\{T-N_n \leq x\}} (e^{T-N_n-x} + (T-s)e^{-x})dx
$$

and we quickly find the process $g_s(\cdot, dx)$

$$
g_s(\cdot, dx) = \mathbb{I}_{\{T-s \leq x < T-N_n\}} \frac{1}{T-s} + \mathbb{I}_{\{T-N_n \leq x\}} \frac{1 + e^{T-N_n-x}}{T-s + e^{T-N_n-x}}
$$

such that $g_s(\cdot, \tau_{N_n})$ is, by Theorem 2, the information drift.

### 5.2. $L = T_n$. The random variable in the previous example satisfies Jacod’s condition as its conditional distribution given $\mathcal{F}_t$ is continuous with respect to its law. The simple example we consider here, $L = T_n$, does not, as confirmed in Mensi and Privault (2003). To an extent, this is to be expected: knowing the time of a jump at time 0 is not compatible with the fact that a local martingale can have a jump of size 1 at that time. The conditional distribution of $T_n$ given $\mathcal{F}_t$ is

$$
P_t(\cdot, dx) = \mathbb{I}_{\{0 \leq T_n \leq t\}} \delta_{T_n}(dx) + \mathbb{I}_{\{t < T_n\}} \mathbb{I}_{\{x \geq t\}} \frac{1}{(n-1 - N_t)!} e^{-(x-t)} dx
$$

and as a result

$$
\mathbb{E}[p_n(x - T_n)|\mathcal{F}_t] = \mathbb{I}_{\{0 \leq T_n \leq t\}} \theta_{t-n}(x - T_n) + \mathbb{I}_{\{t < T_n\}} \int_t^\infty p_n(y-t) \theta_{t-n}(x-y) dy
$$

where $\theta_{t-n}(x) = \frac{\sqrt{\frac{2}{\pi t}}} {\sqrt{2\pi t}}$ and $p_n(x) = \frac{\sqrt{\frac{2}{\pi t}}} {\sqrt{2\pi t}} e^{-x}$. Nualart and Vives (1990) calculated $\Psi_sT_n$ explicitly and as a result we know $\Psi_s(f(T_n))$ equals

$$
f(T_{n-1}) - f(T_n) \text{ if } s < T_{n-1}
$$

$$
f(s) - f(T_n) \text{ if } T_{n-1} \leq s \leq T_n
$$

$$
0 \text{ if } T_n \leq s.
$$

Similarly, one can easily work out that $\Psi_s(f(N_t))$ equals

$$
f(N_{t+1}) - f(N_t) \text{ if } s \leq t
$$

$$
0 \text{ if } s > t.
$$

We are now in a position to calculate $\Psi_s \mathbb{E}[p_n(x - T_n)|\mathcal{F}_t]$. Using relation (13), we derive

$$
\Psi_s(\mathbb{I}_{\{T_n \leq t\}} \theta_{t}(x - T_n)) = (\mathbb{I}_{\{T_{n-1} \leq t\}} \theta_{t}(x - T_{n-1}) - \mathbb{I}_{\{T_n \leq t\}} \theta_{t}(x - T_n)) \mathbb{I}_{\{s < T_{n-1}\}} + \mathbb{I}_{\{s \leq t\}} \theta_{t}(x - s) - \mathbb{I}_{\{T_n \leq t\}} \theta_{t}(x - T_n) \mathbb{I}_{\{T_{n-1} \leq s \leq T_n\}}
$$

and for the second part involving an integral, we notice it is a product of functions of $T_n$ and $N_t$. Lemma 6.1 of Nualart and Vives (1990) provides a product rule for $\Psi$: if $f$ and $g$ are functionals over $\Omega$, then

$$
\Psi_t(f \cdot g) = f \cdot \Psi_t(g) + g \cdot \Psi_t(f) + \Psi_t(f) \cdot \Psi_t(g).
$$
We shall take \( f = I_{\{T_n > t\}} \) and \( g = \int_t^\infty p_{n-1-N_s}(y-t)\theta_\epsilon(x-y)dy \) and it is easy to find

\[
\Psi_s(f) = \left( I_{\{t < T_{n-1}\}} - I_{\{t < T_n\}} \right) I_{\{s < T_{n-1}\}}
+ \left( I_{\{t < s\}} - I_{\{t < T_n\}} \right) I_{\{T_{n-1} \leq s \leq T_n\}}
\]

\[
\Psi_s(g) = \left( \int_t^\infty (p_{n-2-N_s}(y-t) - p_{n-1-N_s}(y-t))\theta_\epsilon(x-y)dy \right) I_{\{s \leq t\}}
\]

and after some algebra and computation, we reach

\[
\Psi_s(f \cdot g) = I_{\{s \leq t\}}\left[ I_{\{t,s < T_{n-1}\}} \int_t^\infty p_{n-2-N_s}(y-t)\theta_\epsilon(x-y)dy
- I_{\{t,s < T_n\}} \int_t^\infty p_{n-1-N_s}(y-t)\theta_\epsilon(x-y)dy \right]
\]

The next step is to calculate the conditional expectation with respect to \( F_s \). Using the conditional law of \( T_n \) given in Equation 12, we work out that

\[
E[\Psi_s(1_{\{T_n \leq t\}}\theta_\epsilon(x - T_n))|F_s] \text{ is equal to }
\]

\[
1_{\{T_{n-1} \leq s \leq T_n\}} \theta_\epsilon(x - s)
- 1_{\{T_n \geq s\}} \int_s^t p_{n-1-N_s}(y - s)\theta_\epsilon(x - y)dy
+ 1_{\{T_{n-1} \geq s\}} \int_s^t p_{n-2-N_s}(y - s)\theta_\epsilon(x - y)dy.
\]

To find the conditional expectation of \( \Psi_s(f \cdot g) \), we need to know the conditional law of \( N_t \) given \( F_s \). It is easy to show that it is

\[
P(N_t = k|F_s) = e^{-(t-s)} \frac{(t - s)^{k-N_s}}{(k-N_s)!} I_{\{k \geq N_s\}}.
\]

Let us find calculate

\[
E[1_{\{t < T_n\}} \int_t^\infty p_{n-1-N_s}(y-t)\theta_\epsilon(x-y)dy|F_s]
\]

and the rest of \( E[\Psi_s(f \cdot g)|F_s] \) can be worked out similarly. After plugging in conditional law (15), we see that expression (16) is equal to

\[
\sum_{k=N_s}^{n-1} e^{-(t-s)} \int_t^\infty \frac{(t - s)^{k-N_s}}{(k-N_s)!} \frac{(y-t)^{n-1-k}}{(n-1-k)!} e^{-(y-t)}\theta_\epsilon(x-y)dy.
\]

Substituting \( K = n - 1 - N_s \) into the summation and an application of the binomial theorem simplifies things further:

\[
\sum_{k=N_s}^{n-1} \frac{(t - s)^{k-N_s}}{(k-N_s)!} \frac{(y-t)^{n-1-k}}{(n-1-k)!} = \frac{1}{K!} \sum_{k=0}^{K} \frac{(t - s)^{K-k}}{K-k} \left( \frac{K}{k} \right) = \frac{(y-s)^K}{K!}.
\]

As a result we can write \( E[\Psi_s(f \cdot g)|F_s] \) as

\[
1_{\{s \leq t\}}[-1_{\{T_n \geq s\}} \int_t^\infty p_{n-1-N_s}(y-s)\theta_\epsilon(x-y)dy
+ 1_{\{T_{n-1} \geq s\}} \int_t^\infty p_{n-2-N_s}(y-s)\theta_\epsilon(x-y)dy]
\]
and after summing the above with \( \mathbb{E}[\Psi_s(\mathbb{1}_{\{T_n\leq t\}}\theta_e(x - T_n))|\mathcal{F}_s] \) we finally have

\[
\mathbb{E}[\Psi_s \mathbb{E}[p_t(x - T_n)|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{1}_{\{T_{n-1}\leq s\leq T_n\}}\theta_e(x - s) \\
+ \mathbb{1}_{\{T_{n-1}\geq s\}} \int_s^\infty p_{n-2-N_e}(y - s)\theta_e(x - y)dy \\
- \mathbb{1}_{\{T_n\geq s\}} \int_s^\infty p_{n-1-N_e}(y - s)\theta_e(x - y)dy.
\]

We are now in a position to take the limit of \( k_n^*(\cdot, dx) \) as \( \epsilon \to 0 \). In doing so we find the measure-valued random process \( k_n^*(\cdot, dx) \) is given by

\[
\mathbb{1}_{\{T_{n-1}\leq s\leq T_n\}} \delta_s(dx) + \mathbb{1}_{\{T_{n-1}\geq s\}} \mathbb{1}_{[s,\infty)}(x)p_{n-2-N_e}(x - s)dx \\
- \mathbb{1}_{\{T_n\geq s\}} \mathbb{1}_{[s,\infty)}(x)p_{n-1-N_e}(x - s)dx.
\]

Thus because \( P_s(\cdot, dx) \) has Dirac measure at \( T_n \) and \( k_n^*(\cdot, dx) \) has one at \( s \), there is no random process \( g_n(\cdot, x) \) such that \( k_n^*(\cdot, dx) = g_n(\cdot, x)P_s(\cdot, dx) \) and there is no information drift.

5.3. \( L = \mathbb{1}_{[u_1, u_2]}(\tilde{N}_T) \). In this example, \( L \) represents the knowledge whether the final value of the compensated Poisson process lies in a given interval. We will require the condition \( u_2 - u_1 > 2 \). Since \( \tilde{N}_T = N_T - T \) and \( N_T \) is a non-negative integer, it is clear that

\[
L = \mathbb{1}_{[u_1, u_2]}(\tilde{N}_T) = \mathbb{1}_{\left[\{u_1 + T\leq N_T \leq u_2 + T\}\right]}
\]

where \( \lfloor x \rfloor = \max\{m \in \mathbb{Z}|m \leq x\} \) and \( \lceil x \rceil = \min\{n \in \mathbb{Z}|n \geq x\} \). The distribution of \( L \) is given by

\[
P(L = 1) = \sum_{k = \lfloor u_1 + T \rfloor}^{\lfloor u_2 + T \rfloor} \frac{T^k}{k!}e^{-T} \\
P(L = 0) = 1 - P(L = 1).
\]

The conditional distribution of \( L \) given \( \mathcal{F}_t \) is found by considering the number of jumps that have occurred by time \( t \), \( N_t \). If \( N_t > \lfloor u_2 + T \rfloor \), \( L \) must be zero at time \( T \) because there have been too many jumps. If \( \lfloor u_1 + T \rfloor \leq N_t \leq \lfloor u_2 + T \rfloor \), \( L \) will take the value 1 if and only if there are between 0 and \( \lfloor u_2 + T \rfloor - N_t \) jumps in the remaining time. If \( N_t < \lfloor u_1 + T \rfloor \), \( L \) will only take the value 1 if there are at least \( \lfloor u_1 + T \rfloor - N_t \) and at most \( \lfloor u_2 + T \rfloor - N_t \) in the time remaining. Using the independent increment property of the Poisson process, we find the conditional distribution of \( L \) given \( \mathcal{F}_t \) to be

\[
P(L = 1|\mathcal{F}_t) = \mathbb{1}_{\{u_1 + T\leq N_t \leq u_2 + T\}} \sum_{k = 0}^{\lfloor u_2 + T \rfloor - N_t} \frac{(T - t)^k}{k!}e^{-(T-t)} \\
+ \mathbb{1}_{\{N_t < \lfloor u_1 + T \rfloor\}} \sum_{k = \lfloor u_1 + T \rfloor - N_t}^{\lfloor u_2 + T \rfloor - N_t} \frac{(T - t)^k}{k!}e^{-(T-t)}
\]

\[
P(L = 0|\mathcal{F}_t) = 1 - P(L = 0|\mathcal{F}_t)
\]

and the conditional law is

\[
P_t(\cdot, dx) = P(L = 1|\mathcal{F}_t)\delta_1(dx) + P(L = 0|\mathcal{F}_t)\delta_0(dx).
\]
It follows that
\[ \Psi_s(\theta_\epsilon(x-L)|F_t) = (\theta_\epsilon(x-1) - \theta_\epsilon(x-0))\Psi_s(\theta(L = 1|F_t)). \]

The only random variable in the conditional distribution is \(N_t\). It is clear that \(\Psi_s(N_t) = N_t + 1 - N_t\) if \(s \leq t\) and is zero otherwise. Using this we find the following relation for \(s \leq t\)
\[
\Psi_s(P(L = 1|F_t)) = \mathbb{I}_{[u_1, T]} - \mathbb{I}_{[u_1 + T, u_2 + T]} \sum_{l=N_t}^{u_2 + T} \frac{(T - t)^{l - N_t}}{(l - N_t)!} e^{-(T-t)}
\]

and fortunately this simplifies to
\[
(19) \quad \Psi_s(P(L = 1|F_t)) = \mathbb{I}_{[u_1, u_1 + T]} + \mathbb{I}_{[u_1 + T, u_2 + T]} - \mathbb{I}_{[u_1 + T, u_2 + T]} \sum_{l=N_t}^{u_2 + T} \frac{(T - t)^{l - N_t}}{(l - N_t)!} e^{-(T-t)}
\]

thanks to our assumption that \(u_2 - u_1 > 2\) which ensures the following ordering of the sum limits:

\[ [u_1 + T] - 1 < [u_1 + T] \leq [u_2 + T] - 1 < [u_2 + T]. \]

Notice that both parts of (19) can be written as \(\mathbb{I}_{(N_t \leq a)} P(N_T = a|F_t)\). Taking the conditional expectation of terms like this with respect to \(F_s\) simplifies matters further:
\[
E[\mathbb{I}_{(N_t \leq a)} P(N_T = a|F_t)|F_s] = E[(1 - \mathbb{I}_{(N_t > a)} P(N_T = a|F_t))|F_s]
\]
\[ = E[P(N_T = a|F_t)|F_s] - E[\mathbb{I}_{(N_t > a)} P(N_T = a|F_t)|F_s]
\]
\[ = P(N_T = a|F_s). \]

It follows that
\[ k_s^* \cdot dx = (\theta_\epsilon(x-1) - \theta_\epsilon(x-0)) P(N_T = [u_1 + T] - 1|F_s) - P(N_T = [u_2 + T]|F_s) \]

and after taking limits as \(\epsilon \to 0\) we find
\[
k_s \cdot dx = [P(N_T = [u_2 + T]|F_s) - P(N_T = [u_1 + T] - 1|F_s)] \delta_0(dx)
\]
\[ + [P(N_T = [u_1 + T] - 1|F_s) - P(N_T = [u_2 + T]|F_s)] \delta_1(dx). \]
Recalling the conditional law of $L$ given in equation (18), we notice that there is a random process $g_s(\cdot, dx)$ given by

$$
g_s(\cdot, dx) = \frac{P(N_T = \lfloor u_2 + T \rfloor | F_s) - P(N_T = \lfloor u_1 + T \rfloor - 1 | F_s)}{1 - P(\lfloor u_1 + T \rfloor \leq N_T \leq \lfloor u_2 + T \rfloor | F_s)} \mathbb{1}_{\{x=0\}}
+ \frac{P(N_T = \lceil u_1 + T \rceil - 1 | F_s) - P(N_T = \lfloor u_2 + T \rfloor | F_s)}{P(\lceil u_1 + T \rceil \leq N_T \leq \lfloor u_2 + T \rfloor | F_s)} \mathbb{1}_{\{x=1\}}$$

such that $g_s(\cdot, L)$ is the information drift.

6. Conclusions

Throughout this article we have used terms like ‘trader’, ‘information’ and ‘additional utility’ to provide an intuitive understanding of our results. In order for these words to carry more import than simply providing analogies for probabilistic entities (a good example being our use of ‘information drift’ for $g_s(\cdot, L)$), there needs to be a viable model for stock prices for which our results apply. Previous results relating to enlargement of filtrations on the Weiner space, as found in Imkeller at al. (2001) and Pikovsky and Karatzas (1996) amongst others, have had an impact on financial mathematics because a geometric Brownian motion is a widely applied model for asset prices.

In order for the results of this paper to have a similar impact, there needs to be a decent model for stock price dynamics generated solely by a compensated Poisson process. Mensi and Privault (2003) consider such a model, where the logarithmic price process is a compensated Poisson process with a drift. However, this is a toy model whose behaviour has little resemblance to price movements in the real-world. It is for this reason, the lack of a decent price model based purely on a compensated Poisson process, that we do not investigate concepts like expected additional logarithmic utility and arbitrage possibilities further, as these only make sense in a financial setting. We have used financial terminology to aid conceptualisation of the results, but we cannot pretend, as yet, to have found results applicable to mathematical finance.

However, we can conclude that this paper has extended the technique of enlargement of filtrations on Poisson space. By showing that the method of Imkeller et al. (2001) can be directly applied using the Poisson space based calculus of variations derived in Privault (1994) and Mensi and Privault (2003), in Theorem 1 we have proven a measure-valued Clark-Ocone formula and in Theorem 2 we have extended Proposition 3 in Mensi and Privault (2003) by actually providing the term in the stochastic integral and sufficient conditions for its existence. In essence, this paper provides the machinery for calculating enlargements of filtration on Poisson space allowing us to calculate two new examples.

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REFERENCES


