Chapter 5

Weighted Least Square

5.1 Weighted Least Square (WLS)

- Model

$$E(Y \mid X = x_i) = \beta' x_i$$

$$Var(Y \mid X = x_i) = \frac{\sigma^2}{w_i}$$

assumed known

- Alternative representation

$$Y = X\beta + e, \qquad Var(e) = \sigma^2 W^{-1} = \sigma^2 \begin{pmatrix} \frac{1}{w_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{w_2} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{w_n} \end{pmatrix}$$

  - Errors are independent but not identically distributed

5.1 Weighted Least Square (WLS)

- Model $E(Y \mid X = x_i) = \beta' x_i, \qquad Var(Y \mid X = x_i) = \dfrac{\sigma^2}{w_i}$ assumed known

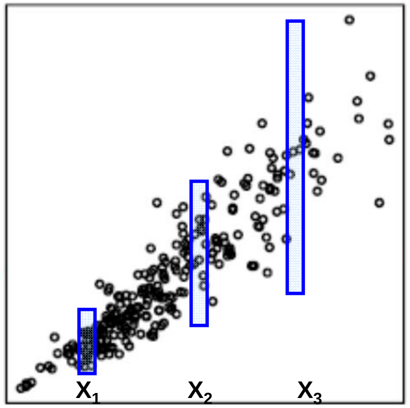
- Examples of known w_i
  - The i-th Observations is an average of n_i variables

    $$Y_i = \frac{z_{i1} + z_{i2} + \ldots + z_{in_i}}{n_i}, \quad Var(Y_i \mid X = x_i) = \frac{Var(z_{i1} \mid X = x_i)}{n_i} = \frac{\sigma^2}{n_i} \Rightarrow w_i = n_i$$

  - The i-th Observations is a sum of n_i variables

    $$Y_i = z_{i1} + z_{i2} + \ldots + z_{in_i}, \quad Var(Y_i \mid X = x_i) = n_i Var(z_{i1} \mid X = x_i) = n_i \sigma^2 \Rightarrow w_i = 1/n_i$$

  - When sample size is large, estimate $Var(Y \mid X = x_i)$ by computing sample variance of the Y with X close to x_i.
  - Subject knowledge…
  - Guess from the scatterplot…



5.1 Estimators for the parameters

- Residual sum of square
  - Standardized by the variance of each observation

$$
\begin{aligned}
\mathrm{RSS}(\beta) &= \sum \frac{(y_i - \hat{y}_i)^2}{Var(y_i)} = \frac{1}{\sigma^2} \sum w_i (y_i - \hat{y}_i)^2 \\
&\propto \sum w_i (y_i - \hat{y}_i)^2 \\
&= \begin{pmatrix} y_1 - \hat{y}_1 & \cdots & y_n - \hat{y}_n \end{pmatrix} \begin{pmatrix} w_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & w_n \end{pmatrix} \begin{pmatrix} y_1 - \hat{y}_1 \\ \vdots \\ y_n - \hat{y}_n \end{pmatrix} \\
&= (Y - X\beta)' W (Y - X\beta)
\end{aligned}
$$

5.1 Estimators for the parameters

- Residual sum of square

$$\sqrt{W} = \begin{pmatrix} \sqrt{w_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sqrt{w_n} \end{pmatrix}, \qquad \sqrt{W} \times \sqrt{W} = W$$

$$\text{RSS}(\beta) = (Y - X\beta)'W(Y - X\beta) = (\sqrt{W}Y - \sqrt{W}X\beta)'(\sqrt{W}Y - \sqrt{W}X\beta)$$

Recall: In multiple linear regression,

$$\hat{\beta} = (X'X)^{-1}X'Y \quad \text{minimizes} \quad (Y - X\beta)'(Y - X\beta)$$

- Therefore, the WLS estimator is

$$\boxed{\hat{\beta}_w = ((\sqrt{W}X)'\sqrt{W}X)^{-1}(\sqrt{W}X)'\sqrt{W}Y = (\mathrm{X'WX})^{-1}\mathrm{X'WY}}$$

5.1 Transforming OLS to WLS

- WLS → OLS by multiplying square root of W

$$Y = X\beta + e, \qquad Var(e) = \sigma^2 W^{-1}$$

$$\Rightarrow \sqrt{W}Y = \sqrt{W}X\beta + \sqrt{W}e$$

$$\Rightarrow Z = M\beta + d, \qquad Var(d) = \sigma^2 I$$

$$\begin{pmatrix} \sqrt{w_1}Y_1 \\ \vdots \\ \sqrt{w_n}Y_n \end{pmatrix} \quad \begin{pmatrix} \sqrt{w_1} & \sqrt{w_1}X_{11} & \sqrt{w_1}X_{12} & \cdots & \sqrt{w_1}X_{1p} \\ \sqrt{w_2} & \sqrt{w_2}X_{21} & \sqrt{w_2}X_{22} & \cdots & \sqrt{w_2}X_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{w_n} & \sqrt{w_n}X_{n1} & \sqrt{w_n}X_{n2} & \cdots & \sqrt{w_n}X_{np} \end{pmatrix} \quad \begin{pmatrix} \sqrt{w_1}e_1 \\ \vdots \\ \sqrt{w_n}e_n \end{pmatrix}$$

WLS summary

- All you need to know
  - Estimators
    $$\hat{\beta} = (M'M)^{-1}M'Z = (X'WX)^{-1}X'WY$$
    $$\sigma^2 = RSS/(n-p-1) = (Z-M\hat{\beta})'(Z-M\hat{\beta})/(n-p-1)$$
  - Distribution of estimators
    $$\hat{\beta} \sim N(\beta, \sigma^2(M'M)^{-1}) \qquad \frac{(n-p-1)\hat{\sigma}^2}{\sigma^2} = \chi^2_{n-p-1}$$
  - F-test
  - T-test
    $$T = \frac{\hat{\beta}_k}{sd(\hat{\beta}_k)} \sim t(n-p-1)$$
    $$F = \frac{(RSS_{NH} - RSS_{AH})/(df_{NH} - df_{AH})}{RSS_{AH}/df_{AH}} =\sim F(df_{NH} - df_{AH}, df_{AH})$$
  - Prediction Interval.
    $$\hat{y}_* \pm t\left(\frac{\alpha}{2}, n-p-1\right)\hat{\sigma}\sqrt{\boldsymbol{1} + \boldsymbol{x}_*'(M'M)^{-1}\boldsymbol{x}_*}, \quad \hat{y}_* = \boldsymbol{x}_*\hat{\beta}$$
  - C.I. for fitted value
    $$\hat{y} \pm t\left(\frac{\alpha}{2}, n-p-1\right)\hat{\sigma}\sqrt{\boldsymbol{x}'(M'M)^{-1}\boldsymbol{x}}, \qquad \hat{y} = \boldsymbol{x}\hat{\beta}$$
  - C.B. for fitted value
    $$\hat{y} \pm \sqrt{(p+1)F(\alpha, p+1, n-p-1)}\hat{\sigma}\sqrt{\boldsymbol{x}'(M'M)^{-1}\boldsymbol{x}}$$
  - Confidence Ellipse
    $$\frac{(\hat{\beta}-\beta)'(M'M)(\hat{\beta}-\beta)}{(p+1)\hat{\sigma}^2} \le F(\alpha, p+1, n-p-1)$$

5.1 Example-Strong interaction data

- y=scattering cross section, x=1/(momentum)$^{0.5}$
  - # Manipulate the data

```
library(alr3); data(physics);x=physics
#[Or x<-read.table("C://physics.txt",header=T)]
x$M=x$x/x$SD
x$z=x$y/x$SD
x$int=1/x$SD
x
```

  - # R command for WLS

```
summary(lm(x$y~x$x,weights=(1/x$SD^2)))
```

  - # What is the difference between the following two commands?

```
summary(lm(x$y~x$x,weights=(1/x$SD^2)))
summary(lm(x$z~x$int+x$M-1))
```

  - # OLS technique for WLS

```
fit1=lm(x$z~x$int-1)
fit2=lm(x$z~x$int+x$M-1)
RSS.H0= sum(fit1$residuals^2)
RSS.HA= sum(fit2$residuals^2)
R.square=1-RSS.HA/RSS.H0
F.stat=(RSS.H0-RSS.HA)/(RSS.HA/fit2$df)
```

TABLE 5.1 The Strong Interaction Data

$x = s^{-1/2}$	y (μb)	SD_i
0.345	367	17
0.287	311	9
0.251	295	9
0.225	268	7
0.207	253	7
0.186	239	6
0.161	220	6
0.132	213	6
0.084	193	5
0.060	192	5

	x	y	SD	M	z	int
1	0.345	367	17	0.02029412	21.58824	0.05882353
2	0.287	311	9	0.03188889	34.55556	0.11111111
3	0.251	295	9	0.02788889	32.77778	0.11111111
4	0.225	268	7	0.03214286	38.28571	0.14285714
5	0.207	253	7	0.02957143	36.14286	0.14285714
6	0.186	239	6	0.03100000	39.83333	0.16666667
7	0.161	220	6	0.02683333	36.66667	0.16666667
8	0.132	213	6	0.02200000	35.50000	0.16666667
9	0.084	193	5	0.01680000	38.60000	0.20000000
10	0.060	192	5	0.01200000	38.40000	0.20000000

TABLE 5.2 WLS Estimates for the Strong Interaction Data

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  148.473      8.079   18.38 7.91e-08
x            530.835     47.550   11.16 3.71e-06

Residual standard error: 1.657 on 8 degrees of freedom
Multiple R-Squared: 0.9397

Analysis of Variance Table
          Df Sum Sq Mean Sq F value    Pr(>F)
x          1 341.99  341.99  124.63 3.710e-06
Residuals  8  21.95    2.74
```