Properties Least Square estimator

- Model

$$Y = X\beta + e, \qquad \mathrm{E}(e) = 0, Var(e) = \sigma^2 I$$

- Least Square Estimator (LSE)

$$\boxed{\hat{\beta} = (X'X)^{-1}X'Y}$$

- Mean of LSE (Unbiasedness: mean of estimate= truth)

$$E(\hat{\beta}) = (X'X)^{-1}X'E(Y) = (X'X)^{-1}X'E(X\beta + e)$$
$$= (X'X)^{-1}X'X\beta = \beta$$

- Variance of LSE

$$Var(\hat{\beta}) = Var[(X'X)^{-1}X'Y] = (X'X)^{-1}X'Var(Y)X(X'X)^{-1}$$
$$= (X'X)^{-1}X'(\sigma^2 I)X(X'X)^{-1} = \sigma^2(X'X)^{-1}(X'X)(X'X)^{-1}$$
$$= \sigma^2(X'X)^{-1}$$

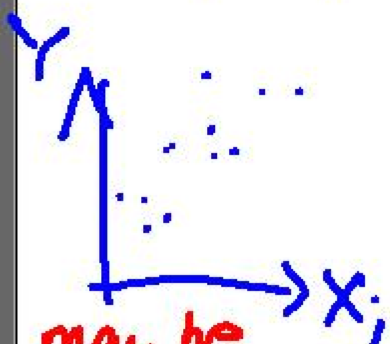
substitute $X = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$

simple linear regression

$$= \sigma^2 \begin{pmatrix} \cdots & \cdots \\ \cdots & \frac{1}{Sxx} \end{pmatrix} \to var(\hat{\beta}_1)$$

- Vertical: Residual of the regression of y on all predictors except x_i
- Horizontal: Residual of the regression of x_i on all other predictors.

$Y \quad X_i \quad X_o$

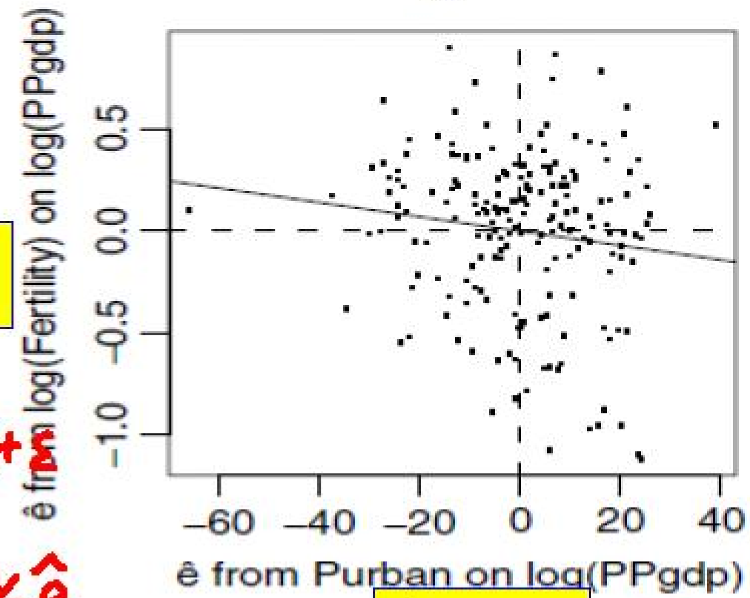


may be affected by X_o

$\hat{e}_{Y|X_o}$

fit $Y = X_o\beta + e$

$\hat{e}_{Y|X_o} = Y - X_o\hat{\beta}$



$\hat{e}_{X_i|X_o}$

$$Y = X_1\beta_1 + \ldots + \underline{X_i\beta_i} + \ldots + X_p\beta_p + e$$

$$\Rightarrow Y = \underline{X_i\beta_i} + X_o\beta_o + e$$

fit $X_i = X_o\alpha + e$

$\hat{e}_{X_i|X_o} = X_i - X_o\hat{\alpha}$

- Properties
  - The slope equal to $\hat{\beta}_i$ in multiple regression.
    - can see the effect of x after adjusted for the effect of other predictors
    - The plot gives more information than the coefficient in multiple regression.
  - May have different magnitude, sign and significance compare to $\hat{\beta}_i$ in

- An interesting observation from a computer experiment
  - Data generation
    - x1=rnorm(100); x2=rnorm(100); e=rnorm(100,0,0.1); y=3*x1+2*x2+e
  - Fit regression
    - Fit1=lm(y~x1+x2); summary(Fit1)

Coefficients:

	Estimate	Std. Error	t value	Pr(>|t|)	
(Intercept)	-0.007537	0.010466	-0.72	0.473	
x1	3.019023	0.010082	299.44	<2e-16	***
x2	1.996044	0.010182	196.04	<2e-16	***

    - Fit2=lm(y~x2); summary(Fit2)

Coefficients:

	Estimate	Std. Error	t value	Pr(>|t|)	
(Intercept)	-0.1800	0.3163	-0.569	0.57	
x2	2.4254	0.3051	7.950	3.24e-12	***

    - Fit3a=lm(y~x1); Fit3b=lm(x2~x1); Fit3c=lm(Fit3a$residual~Fit3b$residual); summary(Fit3c)

Coefficients:

	Estimate	Std. Error	t value	Pr(>|t|)	
(Intercept)	-2.975e-17	1.037e-02	-2.87e-15	1	
Fit3b$residual	1.996e+00	1.013e-02	197.1	<2e-16	***

Y vs X_2

lm(Y~X1) $\downarrow$ $\hat{e}_{Y|X_1}$

lm(X_2~X_1) $\downarrow$ $\hat{e}_{X_2|X_1}$

$\hat{e}_{Y|X_1}$

$\hat{e}_{X_2|X_1}$

lm($\hat{e}_{Y|X_1}$ ~ $\hat{e}_{X_2|X_1}$)

β_2 measures the relationship between y and x2, after adjusting for the effect of x1

Added-Variable plot

- Theory: Why the slope in the added-variable plot = $\hat{\beta}_i$?
- To see why, study the residuals
- The hat matrix H=X(X'X)$^{-1}$X':

$$\hat{e} = Y - \hat{Y} = \left((Y_1 - \hat{Y}_1)\ \ (Y_1 - \hat{Y}_1) \ldots (Y_n - \hat{Y}_n)\right)^T$$
$$= Y - X\hat{\beta}$$
$$= Y - X(X'X)^{-1}X'Y$$
$$= (1 - \underbrace{X(X'X)^{-1}X'}_{H = X(X'X)^{-1}X'})Y$$
$$= (1 - H)Y$$

- When fitting $Y = X\beta + e$, we have

$$\boxed{\hat{e} = (1 - H)Y}$$

$$\boxed{(1 - H)X = X - X\cancel{(X'X)}^{-1}\cancel{X'}X = X - X = 0}$$

- **Idea**
  - Properties $(I-H_{Oth})X_O = 0$

$$\hat{e}_{Y|X_O} = (I-H_{Oth})Y$$

$$\hat{e}_{X_1|X_O} = (I-H_{Oth})X_1$$

  - Adjust for X_O : $Y = X_1\beta_1 + X_O\beta_O + e$

$$\star \Rightarrow (I-H_{Oth})Y = (I-H_{Oth})X_1\beta_1 + \overset{0}{(I-H_{Oth})X_O}\beta_O + (I-H_{Oth})e$$

$$= (I-H_{Oth})X_1\beta_1 + \underline{\quad 0 \quad} + (I-H_{Oth})e$$

$$\Rightarrow \hat{e}_{Y|X_O} = \hat{e}_{X_1|X_O}\beta_1 + \tilde{e}$$

  - where $\tilde{e} = (1-H_{Oth})e$
  - Therefore β_1 is the regression coefficient of $\hat{e}_{Y|X_O}$ against $\hat{e}_{X_1|X_O}$

scalar (1st entry of $\hat{\beta}$)

vector

$$\hat{\beta}_1 = \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{\sum(x_i-\bar{x})^2}$$

from chapter 2

$$\beta_1 = \frac{\sum \hat{e}_{Y|X_O}\hat{e}_{X_1|X_O}}{\sum \hat{e}^2_{X_1|X_O}} = \frac{X_1'(I-H_{Oth})Y}{X_1'(I-H_{Oth})X_1}$$

$$\hat{\beta} = (X'X)^{-1}X'Y$$