3.6 Confidence intervals and bands

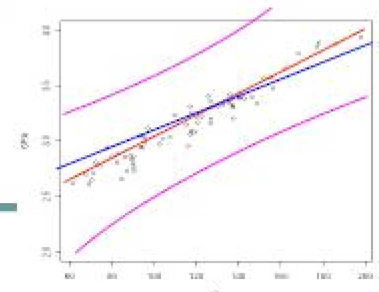
- Confidence band for E(Y|X) (Idea)
  - $$\hat{y} \pm \sqrt{(p+1)F\left(\alpha, p+1, n-p-1\right)}\,\hat{\sigma}\sqrt{\boldsymbol{x}'(X'X)^{-1}\boldsymbol{x}}$$
  - P(For all x, E(Y|X=x) in C.B.)= 1-α
  - $$P\left(x'\beta = \beta_0+\beta_1x_1+\ldots+\beta_px_p \text{ in C.B. for all } x=(1,x_1,\ldots,x_p)'\right)=1-\alpha$$

$\Rightarrow$ One possibility:

$$P(x'\hat{\beta} - \text{Error bound} \le x'\beta \le x'\hat{\beta} + \text{Error bound}, \text{ all } x) = 1-\alpha$$

$$\Rightarrow P(\max_x |x'\beta - x'\hat{\beta}| \le \text{Error bound}) = 1-\alpha$$

$$\Rightarrow Study \ \max_x x'(\beta-\hat{\beta})$$



$$X'(\hat\beta-\beta) = \underbrace{[(X'X)^{-\frac{1}{2}}X]'}_{x}\underbrace{[(X'X)^{\frac{1}{2}}(\hat\beta-\beta)]}_{y}$$

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- Cauchy Schwartz Inequality

$$\boxed{(x'y)^2 \le (x'x)(y'y)}$$

- Proof of C.B. --- A Super trick! (optional)

Put $x=(X'X)^{-\frac{1}{2}}\mathrm{x},\quad y=(X'X)^{\frac{1}{2}}(\hat\beta-\beta)$, we have

$$\Rightarrow \left(\mathrm{x}'(\hat\beta-\beta)\right)^2 \le \left(\mathrm{x}'(X'X)^{-1}\mathrm{x}\right)\left((\hat\beta-\beta)'(X'X)(\hat\beta-\beta)\right) \quad \text{for any x,}$$

$$\Rightarrow \max_{\mathrm{x}} \frac{\left(\mathrm{x}'(\hat\beta-\beta)\right)^2}{\mathrm{x}'(X'X)^{-1}\mathrm{x}} = (\hat\beta-\beta)'(X'X)(\hat\beta-\beta)$$

$$\Rightarrow P\left\{\max_{\mathrm{x}} \frac{\left(\mathrm{x}'(\hat\beta-\beta)\right)^2}{(p+1)\hat\sigma^2\mathrm{x}'(X'X)^{-1}\mathrm{x}} \le F^*\right\} = P\left\{\frac{(\hat\beta-\beta)'(X'X)(\hat\beta-\beta)}{(p+1)\hat\sigma^2} \le F^*\right\} = 1-\alpha$$

$$\Rightarrow P\left\{\mathrm{x}'\beta \in \left[\mathrm{x}'\hat\beta \pm \sqrt{(p+1)F^*}\,\hat\sigma\sqrt{\mathrm{x}'(X'X)^{-1}\mathrm{x}}\right] \text{ for any x}\right\} = 1-\alpha$$

$$F^* = F(\alpha, p+1, n-p-1)$$