$$
\begin{array}{ccccc}
Y & \hat{Y} & \bar{Y} & e & \hat{e} \\
| & | & | & | & | \\
N(X\beta, \sigma^2) & HY & JY & N(0, \sigma^2 I) & (I-H)Y
\end{array}
$$

$\hat{e} = Y - \hat{Y}$

$$
J = \frac{1}{n}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}_{n\times n}
$$

$$
\boxed{JH = J = HJ}
$$

$$
JJJJJ = J
$$

$$
\boxed{JJ = J}
$$

$$
\frac{1}{n^2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} n & n & n & n \\ n & n & n & n \\ n & n & n & n \end{pmatrix}\frac{1}{n^2} = \frac{1}{n}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = J
$$

$$
\mathrm{Cov}(Y, \hat{e}) = \mathrm{Cov}(Y, Y)(I-H)
$$

$$
\mathrm{Cov}(Y, \hat{Y}) = \mathrm{Cov}(Y, Y)H
$$

$$
\begin{aligned}
\mathrm{Cov}(\bar{Y}, \hat{e}) &= J\,\mathrm{Cov}(Y, Y)(I-H) \\
&= \sigma^2 J(I-H) \\
&= \sigma^2 (J - \underbrace{JH}) = 0
\end{aligned}
$$

Sum of col of $H = 1$

$$
\frac{1}{n}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} H = \frac{1}{n}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = J
$$

Sum of Squares

Ch3 : $\underset{\text{TSS}}{\sum (Y_i - \bar{Y})^2} \overset{\text{want}}{=} \underset{\text{RSS}}{\sum (Y_i - \hat{Y}_i)^2} + \underset{\text{SSreg}}{\sum (\hat{Y}_i - \bar{Y})^2}$

$\bar{Y}$ ← scalar; $\hat{Y}_i = X_i \hat{\beta}$; $\bar{Y}$ ← scalar

Pf $(Y - \bar{Y})'(Y - \bar{Y}) \overset{?}{=} (Y - \hat{Y})'(Y - \hat{Y}) + (\hat{Y} - \bar{Y})'(\hat{Y} - \bar{Y})$

$\bar{Y} = \begin{pmatrix} \bar{Y} \\ \vdots \\ \bar{Y} \end{pmatrix}_{n \times n?}$

$H = X(X'X)^{-1}X' \quad J = \frac{1}{n}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \Rightarrow H = H, J' = J$

$Y'(I - J)(I - J)Y \overset{?}{=} Y'(I - H)(I - H)Y + Y'(H - J)(H - J)Y$

$Y'(I - J - J + \underbrace{JJ}_{J})Y \overset{?}{=} Y'(I - H - H + \underbrace{HH}_{H})Y + Y'(\underbrace{HH}_{H} - \underbrace{JH}_{J} - \underbrace{HJ}_{J} + \underbrace{JJ}_{J})Y$

$\text{TSS} \to Y'(I - J)Y = \underset{\text{RSS}}{Y'(I - H)Y} + \underset{\text{SSreg}}{Y'(H - J)Y}$

$Y'(I - H + H - J)Y \overset{\checkmark}{=} Y'(I - H)Y + Y'(H - J)Y$

e.g. $E\left(\sum_{i=1}^{n} (Y_i - \hat{Y}_i)(Y_i - \bar{Y}) \right)$

model you used

$\sum a_i b_i = (a_1 \cdots a_n)\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

$$\boxed{\begin{array}{l} \text{True: } Y = \underset{n\times(p_1+1)}{X_1}\beta + e \\ \text{use: } Y = \underset{n\times(p_2+1)}{X_2}\beta + e \end{array}}$$

$= E\left(Y'(I-H_2)(I-J)Y \right)$

$= E\left(Y'(I - H_2 - \cancel{J} + \overbrace{\cancel{HJ}}^{J})Y \right) = E\left(Y'(I-H)Y \right)$

$H_2 = X_2(X_2'X_2)^{-1}X_2'$

$n\cdot\cdot$ $^{-1}$

$= E\left(tr\left(Y'(I-H_2)Y\right)\right) = E\left(tr\left((I-H)YY'\right)\right)$

$= tr\left[(I-H_2)E(YY')\right]$

(True model)

e.g. $\cdot\ Y = X\beta + e \quad e \sim N(0, \sigma^2 I)$

$\cdot\ Y = \mu + e$

$= tr\left[(I-H_2)\cdot E\left((X_1\beta+e)(X_1\beta+e)'\right)\right]$

$= tr\left[(I-H_2)\cdot(X_1\beta\beta'X_1' + \sigma^2 I)\right] = tr\,\underbrace{\overbrace{(I-H_2)X_1\beta}^{n\times 1}\overbrace{\beta'X_1'}^{1\times n}}_{n\times n} + \sigma^2 tr(I-H_2)$

$tr(I) = n$, $tr(H_2) = tr\left((X_2'X_2)^{-1}X_2'X_2\right) = p_2+1$

$= \underbrace{\beta'X_1'(I-H_2)X_1\beta}_{\text{scalar}} + \sigma^2(n - p_2 - 1)$

8.1.3. Residuals when the model is correct

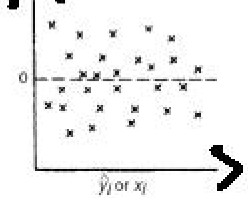
- Look at residual plot
  - y-axis = residuals
  - x-axis = term / combination of terms / fitted values

$e \sim N(0, \sigma^2 I)$

$\hat{e} \sim N(0, \sigma^2 (I - H))$

- Let U be any combinations of terms, we expect the Null Plot
  - $E(\hat{e}_i \mid U) = 0$
    - mean level = 0
  - $Var(\hat{e}_i \mid U) = \sigma^2 (1 - h_{ii})$
    - variance is roughly constant (since h_{ii} are usually small)
    - The point with high leverage ($h_{ii} \sim 1$) have small variance

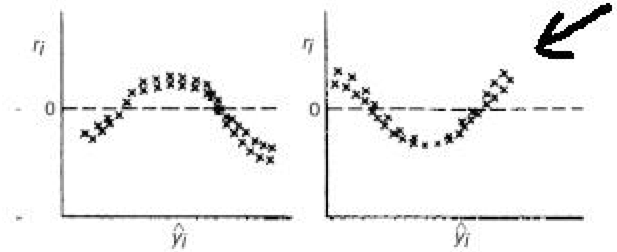
$\hat{e}_i$



$X_1 / X_2 \cdots / X_p / \hat{Y}$

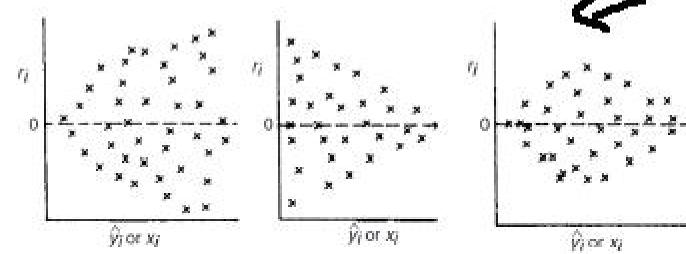
8.1.4. Residuals when the model is NOT correct

- Residual plot far from the Null Plot
  - Mean level not 0



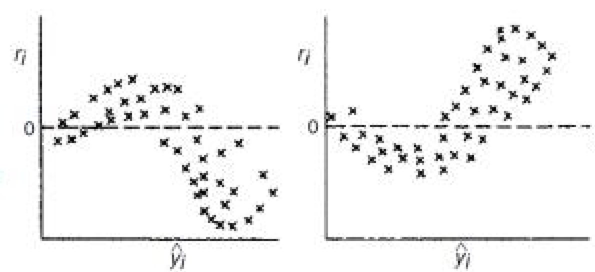
transformation

  - Variance not constant



weighted least square

  - Both



both