8.1. Residuals

$$H = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & \cdots & \cdots & h_{nn} \end{pmatrix} \qquad X = \begin{pmatrix} 1 & x_{11} & \cdots & x_{p1} \\ 1 & x_{12} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{1n} & \cdots & x_{pn} \end{pmatrix}$$

$n \cdot (p+1)$

- More properties of H
  - tr(H)= $\sum_{i=1}^{n} h_{ii}$=p+1

$tr(BA) = tr(AB)$

$$tr(H) = tr(X(X'X)^{-1}X') = tr((X'X)^{-1}\underbrace{X'X}_{(p+1)(p+1)}) = tr(I_{p+1}) = p+1$$

  - $\boxed{\sum_{i=1}^{n} h_{ji} = 1,}$ all j

$$\begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \vdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & \cdots & \cdots & h_{nn} \end{pmatrix} \rightarrow \text{Sum =1}$$

    - Proof:
      - Check the first column of $HX = X$

$X(\cancel{X'X})^{-1}\cancel{X'X} = X$

$\begin{pmatrix} h_{11} & h_{12} & \cdots \\ h_{n1} & \cdots & h_{nn} \end{pmatrix} \begin{pmatrix} 1 & \cdots \\ \vdots & \vdots \\ 1 & \cdots \end{pmatrix} = \begin{pmatrix} 1 & \cdots \\ \vdots & \vdots \\ 1 & \cdots \end{pmatrix}$

$\begin{pmatrix} \sum_{i=1}^{n} h_{1i} & \cdots \\ \sum_{i=1}^{n} h_{2i} & \cdots \\ \vdots & \end{pmatrix}$

  - $\sum_{i=1}^{n} h_{ij} = 1,$ all j

$$\begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \vdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & \cdots & \cdots & h_{nn} \end{pmatrix} \rightarrow \text{Sum =1}$$

    - Proof:
      - Check the first row of $X'H = X'$ or use symmetry of H

- **More properties of H**

$$H = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & \cdots & \cdots & h_{nn} \end{pmatrix} \qquad X = \begin{pmatrix} 1 & x_{11} & \cdots & x_{p1} \\ 1 & x_{12} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{1n} & \cdots & x_{pn} \end{pmatrix}$$

  - tr(H)= $\sum_{i=1}^{n} h_{ii}$=p+1

$$tr(H) = tr(X(X'X)^{-1}X') = tr((X'X)^{-1}X'X) = tr(I_{p+1}) = p+1$$

  - $\sum_{i=1}^{n} h_{ji} = 1, \quad \text{all j}$

$$\begin{pmatrix} \boxed{h_{11} \quad h_{12} \quad \cdots \quad h_{1n}} \\ \boxed{h_{21} \quad h_{22} \quad \vdots \quad h_{2n}} \\ \vdots \quad \vdots \quad \ddots \quad \vdots \\ \boxed{h_{n1} \quad \cdots \quad \cdots \quad h_{nn}} \end{pmatrix} \rightarrow \boxed{\textbf{Sum =1}}$$

    - Proof:
      - Check the first column of $HX = X$

  - $\sum_{i=1}^{n} h_{ij} = 1, \quad \text{all j}$

$$\begin{pmatrix} \boxed{\begin{matrix} h_{11} \\ h_{21} \\ \vdots \\ h_{n1} \end{matrix}} & \boxed{\begin{matrix} h_{12} \\ h_{22} \\ \vdots \\ \cdots \end{matrix}} & \begin{matrix} \cdots \\ \vdots \\ \ddots \\ \cdots \end{matrix} & \boxed{\begin{matrix} h_{1n} \\ h_{2n} \\ \vdots \\ h_{nn} \end{matrix}} \end{pmatrix} \rightarrow \boxed{\textbf{Sum =1}}$$

    - Proof:
      - Check the first row of $\underline{X'H} = \underline{X'}$, or use symmetry of H

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ & \vdots & & \vdots & \end{pmatrix} H = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

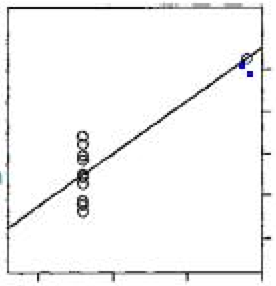
- **Remark:**
  - Another use of H → Diagnostic check for leverage

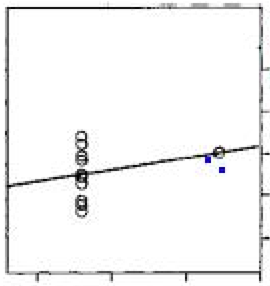
- The Leverage: h_{ii} = The (i,i)-th entry of H
  - meaning: $\begin{pmatrix}\hat{Y}_1\\ \vdots \\ \hat{Y}_n\end{pmatrix} = \hat{Y} = HY = X\hat{\beta} = \underbrace{X(X'X)^{-1}X'}_{H}\,Y$

    i th entry

    $$\Rightarrow \hat{Y}_i = \sum_{k=1}^{n} h_{ik} Y_k = h_{ii} Y_i + \sum_{k \neq i}^{n} h_{ik} Y_k$$

    i th entry of HY
  - As h_{ii} approaches 1, $\hat{Y}_i$ get closer to Y_i
    - h_{ii} is pulling $\hat{Y}_i$ towards Y_i, giving the name Leverage
    - Be careful of leverage point if h_{ii}~1





8.1. Residuals

- With properties of H, we can study property of $\hat{e}$

$$\hat{e} = Y - X\hat{\beta} = Y - X(X'X)^{-1}X'Y = (1-H)Y = \underbrace{(1-H)}_{\text{involve X only, non random}}e$$

- Probabilistic properties of $\hat{e}$
  - Expectation

$$E(\hat{e}) = (I-H)\underbrace{E(e)}_{0} = 0$$

  - Variance

Var(AX) = A Var(X) A'

$$Var(\hat{e}) = (I-H)Var(e)(I-H)'$$
$$= \sigma^2(I-H)I(I-H) \quad \because H \text{ is symmetric}$$
$$= \sigma^2(I-H) \leftarrow HH=H$$

$$\begin{pmatrix} \ddots & & \\ & Var(\hat{e}_i) & \\ & & \ddots \end{pmatrix} \qquad \sigma^2\begin{pmatrix} 1-h_{11} & & \\ & 1-h_{ii} & \\ & & \ddots \end{pmatrix}$$

  - The higher the leverage, the smaller the variance of $\hat{e}_i$

- What are the difference between them?

residual → $\hat{e}^T \cdot 1 = 1^T \cdot \hat{e} = \sum_{i=1}^{n} \hat{e}_i = 0$

$e^T \cdot 1 = 1^T \cdot e = \sum e_i \neq 0$

$RSS = \sum_{i=1}^{n} (Y_i - X_i\beta)^2$

$\frac{\partial RSS}{\partial \beta_0} = -2\sum^{n} \underbrace{(Y_i - X_i\beta)}_{\hat{e}} = 0$

error → $E(e) = 0, \quad Var(e) = \sigma^2 I$

$e \sim N(0, \sigma^2 I)$

off diagonal = 0

⇒ e_i are iid

residual → $E(\hat{e}) = 0, \quad Var(\hat{e}) = \sigma^2 (I - H)$

off diagonal ≠ 0

⇒ $\hat{e}_i$ are correlated

Powerful calculations using H

- $Cov(\hat{e},\hat{Y}) = Cov(\underline{(I-H)Y},\underline{HY}) = E([(I-H)(Y-E(Y)][H(Y-E(Y))]')$

  $= (I-H)\underbrace{E([Y-E(Y)][Y-E(Y)]')}_{Var(Y)}H' = (I-H)\sigma^2 I \cdot H' = \sigma^2 \underbrace{(I-H)H}_{0} = 0$

- $Cov(\hat{e},Y) = Cov(\underline{(I-H)}Y,Y)$

  $= (I-H)\sigma^2 I = \sigma^2 (I-H) \neq 0$

- $Cov(e,Y) = ?$

- $Cov(e,\hat{Y}) = ?$

$$\boxed{\begin{aligned} &Cov(AX,BY) \\ &= A\,Cov(X,Y)\,B' \end{aligned}}$$

$e \sim N(0,\sigma^2 I)$

$E(e) = 0$

$Var(e) = \sigma^2 I$

$E(Y) = X\beta$

$Var(Y) = \sigma^2 I$

Powerful calculations using H

$\sum a_i^2 = (a_1 \cdots a_n) \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

- $$\sum (Y_i - \hat{Y}_i)^2 = (Y - \hat{Y})'(Y - \hat{Y}) = [(I-H)Y]'[(I-H)Y] = Y'(I-H)Y$$

$Y'(I-H)'(I-H)Y$

- $$\begin{pmatrix} \bar{Y} \\ \vdots \\ \bar{Y} \end{pmatrix} = \frac{1}{n} \underline{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix}} \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} \overset{def}{=} \frac{1}{n} JY$$

$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

- $JJ = ? \quad HJ = ? \quad JH = ?$
- $$\sum (Y_i - \bar{Y})^2 = \left(Y - \frac{1}{n}JY\right)'\left(Y - \frac{1}{n}JY\right) = Y'\left(I - \frac{1}{n}J\right)'\left(I - \frac{1}{n}J\right)Y = Y'\left(I - \frac{1}{n}J\right)Y$$
- $$\sum (\hat{Y}_i - \bar{Y})^2 = \left(HY - \frac{1}{n}JY\right)'\left(HY - \frac{1}{n}JY\right) = Y'\left(H - \frac{1}{n}J\right)'\left(H - \frac{1}{n}J\right)Y = Y'\left(H - \frac{1}{n}J\right)Y$$