$r = 0.928$, $S_Y = 2.507$

$\sum x_i = 14.7$, $\sum x_i^2 = 29.15$, $\sum y_i = 40$, $\sum y_i^2 = 244$, $\sum x_i y_i = 82.5$.

$$S_{YY} = \sum y_i^2 - \frac{(\sum y_i)^2}{n} = 244 - \frac{40^2}{8} = 44, \quad S_{XX} = \sum x_i^2 - \frac{(\sum x_i)^2}{n} = 29.15 - \frac{(14.7)^2}{8} = 2.13875.$$

$$RSS = S_{YY} - \frac{S_{XY}^2}{S_{XX}} = 6.1274$$

$$S_{XY} = \sum x_i y_i - \frac{(\sum x_i)(\sum y_i)}{n} = 82.5 - \frac{(14.7)(40)}{8} = 9.$$

ANOVA ~~df~~ ~~SS~~

Source	df	SS	MS	F
Regression	1	37.8726	37.8726	37.08637
Residuals	6	6.1274	1.0212	
Total	7	44		

$T.S. = 37.0864 > F_{1,6,0.05}$.

$t_{6,0.025} = 2.447$, $F_{1,6,0.05} = 2.447^2 = 5.9878$.

$\therefore$ Dependence between X and Y.

$y \sim N(\mu, \sigma^2)$

$y = \beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p + e$

a). $\hat{y} = X\hat{\beta} = X(X^TX)^{-1}X^TY$

$$Cov\left(\frac{1}{n}11^TY, X(X^TX)^{-1}X^TY\right) = Cov\left(\frac{J}{n}Y, HY\right)$$
$$= \frac{J}{n}Cov(Y,Y)H^T$$
$$= \sigma^2\frac{J}{n}H$$
$$= \frac{\sigma^2 J}{n}$$

b). $$E\left[\sum_{i=1}^n(\hat{y}_i - \bar{y})^2\right] = E\left[Y^T(H-\frac{J}{n})(H-\frac{J}{n})Y\right]$$
$$= E\left[Y^T(H-\frac{J}{n})Y\right]$$
$$= E\left[tr\left[Y^T(H-\frac{J}{n})Y\right]\right]$$
$$= E\left[tr\left[(H-\frac{J}{n})YY^T\right]\right]$$
$$= tr\left[(H-\frac{J}{n})(\sigma^2I+\mu^2J)\right]$$
$$= tr\left[\sigma^2(H-\frac{J}{n})+\mu^2(JH-\frac{J^2}{n})\right]$$
$$= \sigma^2 tr(H-\frac{J}{n})$$
$$= \sigma^2(p+1-1) = \sigma^2 p.$$

$E(YY^T) = Var(Y) + E(Y)E(Y)^T$
$= \sigma^2 I + \mu 1(\mu 1)^T$
$= \sigma^2 I + J\mu^2$

第 3. 题
(答题不得寫在紅线外)

$n=10$, $BIC = n\log\left(\frac{RSS}{n}\right) + p_c \log(n)$

RSS	$p_c\,(p+1)$	BIC
18.8	3	13.22
14.3	4	12.79
4.8	5	4.17
2.4.	5	-2.76.

Model 4.

AIC, Mallow's C_p. Statistics.

$$Y = X\beta + e, \quad e \sim N(0, \sigma^2)$$

$$\hat{\beta} = (X^TX + kI)^{-1} X^TY,$$

(a). $E(\hat{\beta}) = E[(X^TX+kI)^{-1}X^TY]$

$$= (X^TX+kI)^{-1}X^TX\beta$$

$$= (X^TX+kI)^{-1}(X^TX + kI - kI)\beta$$

$$= [I - k(X^TX+kI)^{-1}]\beta$$

$$= \beta - k(X^TX+kI)^{-1}\beta.$$

If $k=0 \Rightarrow \hat{\beta}$ is unbiased.

(b). $Var(\hat{\beta}) = (X^TX+kI)^{-1}X^T\,Var(Y)\,X(X^TX+kI)^{-1}$

$$= \sigma^2(X^TX+kI)^{-1}(X^TX)(X^TX+kI)^{-1}$$

$$= \sigma^2(X^TX+kI)^{-1} - k\sigma^2(X^TX+kI)^{-2}.$$

(c). Multicollinearity.

$\rightarrow (X^TX+kI)^{-1}$. $\cancel{Var(\hat{\beta}) = (X^TX+kI)^{-1}\sigma^2}$.

k makes (X^TX+kI) invertible even though X^TX is invertible

(a) $Y = \beta_0 + X_1\beta_1 + X_2\beta_2 + C\beta_3 + e, \quad e \sim N(0, \sigma^2 W)$

$$C = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad W = \begin{pmatrix} \frac{1}{4} & & & & & & 0 \\ & \frac{1}{6} & & & & & \\ & & \frac{1}{10} & & & & \\ & & & \frac{1}{5} & & & \\ & & & & \frac{1}{9} & & \\ & & & & & 1 & \\ 0 & & & & & & \frac{1}{4} \end{pmatrix}$$

(b)
$$X = \begin{pmatrix} 1 & 60 & 170 & 1 \\ 1 & 45 & 160 & 0 \\ 1 & 90 & 190 & 1 \\ 1 & 65 & 175 & 1 \\ 1 & 55 & 170 & 0 \\ 1 & 70 & 180 & 1 \\ 1 & 60 & 155 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 7 \\ 6 \\ 9 \\ 7.5 \\ 6.5 \\ 8 \\ 4 \end{pmatrix}, \quad W^{-1} = \begin{pmatrix} 4 & & & & & & 0 \\ & 6 & & & & & \\ & & 10 & & & & \\ & & & 5 & & & \\ & & & & 9 & & \\ & & & & & 1 & \\ 0 & & & & & & 4 \end{pmatrix}$$

$$\hat{\beta} = (X^T W^{-1} X)^{-1} X^T W^{-1} Y$$

$$X^T W^{-1} X = \begin{pmatrix} 39 & 2540 & 6745 & 20 \\ 2540 & 175200 & 445825 & 1535 \\ 6745 & 445825 & 1171925 & 3635 \\ 20 & 1535 & 3635 & 20 \end{pmatrix}$$

$$X^T W^{-1} Y = \begin{pmatrix} 274 \\ 18575 \\ 48047.5 \\ 163.5 \end{pmatrix}$$

(c). $H_0: E(Y|X) = \beta_0$

$H_1: E(Y|X) = \beta_0 + C\beta_3$

$$X = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} \sim \end{pmatrix}$$

$$W^{-\frac{1}{2}} Y = W^{-\frac{1}{2}} X\beta + W^{-\frac{1}{2}} e$$

$$Z = M\beta + d, \quad E(d) = 0, \quad Var(d) = \sigma^2 I$$

$RSS_1 = Z^T(I - H)Z = 32.2428, \quad df_1 = 7 - 2 = 5$

$RSS_0 = Z^T(I - \frac{J}{n})Z = 300.8606, \quad df_0 = 7 - 1 = 6$

$$\therefore F = \frac{(300.8606 - 32.2428)/(6-5)}{32.2428/5} = 41.6555$$

$F_{0.05, 1, 5} = 6.6079$

$\therefore$ H_0 is Rejected

第 5 題

(答題不得寫在紅線外)

第 頁

(d)

$\frac{\partial \hat{Y}}{\partial X_1} = 0$

$\hat{X}_1 = 180,$

$\frac{\partial \hat{Y}}{\partial X_2} = 0$

$\hat{X}_2 = 75,$

$\frac{\partial \hat{Y}}{\partial C} = 2.4 > 0.$

$\therefore$ A man with 180kg of weight and 75cm of height has the highest running speed on average.

$$X_i = x_i + a_i, \quad E(a_i)=0, \quad Var(a_i)=\sigma_a^2.$$

$$y_i = \beta_0 + \beta_1 x_i + e_i, \quad Var(e_i) = \sigma^2, \quad E(e_i)=0$$

(a)

$$y_i = \beta_0 + \beta_1 X_i + \gamma_i.$$

$$= \beta_0 + \beta_1(x_i + a_i) + \gamma_i$$

5

$$= \beta_0 + \beta_1 x_i + \beta_1 a_i + \gamma_i$$

$$\therefore \quad e_i = \beta_1 a_i + \gamma_i.$$

$$\gamma_i = e_i - \beta_1 a_i.$$

$$E(\gamma_i) = E(e_i - \beta_1 a_i)$$

$$= 0$$

$$Var(\gamma_i) = Var(e_i - \beta_1 a_i)$$

$$= Var(e_i) + \beta_1^2 Var(a_i)$$

$$= \sigma^2 + \beta_1^2 \sigma_a^2.$$

(b) $$Cov(X_i, \gamma_i) = Cov(x_i + a_i, e_i - \beta_1 a_i)$$

$$= -\beta_1 \sigma_a^2$$

errors and regressor should be uncorrelated.

$$\cancel{c)\ \hat{\beta} \to \frac{Cov(X,Y)}{Var(X)} = \frac{\quad}{\sigma_a^2}}$$

第 6 題

（答題不得寫在紅綫外）

第

(c). $\hat{\beta}_1 \xrightarrow{p} \dfrac{Cov(X,Y)}{Var(X)} = \dfrac{\beta_1 \sigma_x^2}{\sigma_x^2 + \sigma_a^2}$ where $\sigma_x^2 = \lim_{n\to\infty} \frac{1}{n} \sum (x_i - \bar{x})^2$

$\therefore$ $\hat{\beta}_1$ is biased.

第 7 題 $3+\frac{9}{4}$
（答題不得寫在紅綫外） 第 頁

(a)

$$Y=(2,6,0,9,7,9)^T,\quad X=\begin{pmatrix}1&1&2\\1&-1&1\\1&1&3\\1&-1&4\\1&1&-5\\1&-1&-5\end{pmatrix}$$

$$X^TX=\begin{pmatrix}6&0&0\\0&6&0\\0&0&80\end{pmatrix}\quad (X^TX)^{-1}=\begin{pmatrix}1/6&0&0\\0&1/6&0\\0&0&1/80\end{pmatrix}$$

$$X^TY=\begin{pmatrix}33\\-15\\-36\end{pmatrix},\quad \hat\beta=\begin{pmatrix}11/2\\-5/2\\-9/20\end{pmatrix}$$

$$\hat Y=\begin{pmatrix}21/10\\151/20\\33/20\\31/5\\21/4\\41/4\end{pmatrix},\quad \hat e=\begin{pmatrix}-0.1\\-1.55\\-1.65\\2.8\\1.75\\-1.25\end{pmatrix},\quad \hat\sigma^2=\frac{\hat e^T\hat e}{6-3}=\frac{88}{15}=5.8667$$

(b) $x_*=\begin{pmatrix}1\\-1\\3\end{pmatrix}$,

$$x_*^T(X^TX)^{-1}x_*=107/240$$

$$\text{sepred}(\hat y_*|x_*)=\sqrt{\hat\sigma^2(1+107/240)}=2.9124$$

$$\hat y_*|x_*=133/20,\quad \cancel{t_{0.025}}\ t_{0.005,3}=4.032$$

$$=[-5.0928,\ 18.3928]$$

(c) Test $H_0: \beta_1=\beta_2$ vs. $H_0: \beta_1\neq\beta_2$

$\Rightarrow H_0: \beta_1-\beta_2=0$ vs $H_A: \beta_1-\beta_2\neq 0$.

Since $\hat\beta_1-\hat\beta_2$, $\widehat{Var}(\hat\beta_1-\hat\beta_2)=\widehat{Var}(\hat\beta_1)+Var(\hat\beta_2)-2Cov(\hat\beta_1,\hat\beta_2)$

$$T.S.=\frac{\hat\beta_1-\hat\beta_2-0}{\sqrt{\widehat{Var}(\hat\beta_1-\hat\beta_2)}}\sim t_3$$