STAT3008 Exercise 6 Solutions

(2011-2012 2nd Semester)

Q1.

ANOVA Table

Source	SS	df	MS	F
Regression	1848.76	1	1848.76	69.2224
Residuals	480.7357	18	26.7075	
Total	2329.4957	19		

Coefficient Table

Variable	Coeff.	se	t	p-value
Constant	-23.4325	12.74	-1.8393	0.0824
X	1.2713	0.1528	8.32	1.3956e-7
n=20	$R^2 = 0.7936$	$\hat{\sigma} = 5.1679$		

For β_0,

$$t = \frac{-23.4325}{12.74} = -1.8393$$

$$p - value = 0.0824$$

$$n - 2 = 18 \Rightarrow n = 20$$

For β_1,

$$\hat{\beta}_1 = t \times se(\hat{\beta}_1) = 8.32 \times 0.1528 = 1.2713$$

$$p - value = 2 \times \Pr(t_{18} > 8.32) = 1.3956 \times 10^{-7}$$

Obviously, this is a simple linear regression model.

Thus, $F = t^2 = 8.32^2 = 69.2224$ and $df_{reg} = 1$.

$$MS_{reg} = \frac{SS_{reg}}{df_{reg}} = 1848.76$$

$$F = \frac{MS_{reg}}{MS_{residual}} \Rightarrow MS_{residual} = \frac{MS_{reg}}{F} = \frac{1848.76}{69.2224} = 26.7075$$

$$df_{residual} = 20 - 2 = 18$$

$$SS_{residual} = df_{residual} \times MS_{residual} = 480.7357$$

$$SS_{Total} = SS_{residual} + SS_{reg} = 2329.4957$$

$$df_{Total} = df_{residual} + df_{reg} = 19$$

$$\widehat{\sigma} = \sqrt{MS_{residual}} = 5.1679$$

$$R^2 = \frac{SS_{reg}}{SS_{Total}} = 0.7936$$

Q2.

(i)

$$H_0 : E(Y \mid X) = \beta_0 + \beta_2 X_2$$

$$H_1 : E(Y \mid X) = \beta_0 + \beta_2 X_2 E(Y \mid X) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$

Under H_1, $X = \begin{pmatrix} 1 & X_1 & X_2 & X_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 10 & 2 \\ 1 & 1 & 7 & 6 \\ 1 & 3 & 6 & 0 \\ 1 & 3 & 3 & 9 \\ 1 & 5 & 2 & 7 \\ 1 & 5 & -1 & 9 \\ 1 & 7 & -2 & 4 \\ 1 & 7 & -5 & 2 \end{pmatrix}$

$$SSE_{H_1} = Y^T (I - H) T = Y^T (I - X(X^T X)^{-1} X^T) Y = 2.2742$$

$$df_{H_1} = n - (p+1) = 8 - 4 = 4$$

Under H_0, $X = \begin{pmatrix} 1 & X_2 \end{pmatrix} = \begin{pmatrix} 1 & 10 \\ 1 & 7 \\ 1 & 6 \\ 1 & 3 \\ 1 & 2 \\ 1 & -1 \\ 1 & -2 \\ 1 & -5 \end{pmatrix}$

$$SSE_{H_0} = Y^T (I - H) T = Y^T (I - X(X^T X)^{-1} X^T) Y = 656.1994$$

$$df_{H_0} = n - (p+1) = 8 - 2 = 6$$

$$F_{(0.05,2,4)} = 6.9443$$

$$Obs.T.S. = \frac{\dfrac{SSE_{H_0} - SSE_{H_1}}{df_{H_0} - df_{H_1}}}{\dfrac{SSE_{H_1}}{df_{H_1}}} = 575.0883 > 6.9443$$

Therefore, H_0 is rejected at 0.05 significance level.

(ii)

$$E(Y \mid X) = \beta_0 + \beta_1(X_1 + X_2) + \beta_2 X_2 + \beta_3 X_3$$

$$X = \begin{pmatrix} 1 & X_1 + X_2 & X_2 & X_3 \end{pmatrix} = \begin{pmatrix} 1 & 11 & 10 & 2 \\ 1 & 18 & 7 & 6 \\ 1 & 9 & 6 & 0 \\ 1 & 6 & 3 & 9 \\ 1 & 7 & 2 & 7 \\ 1 & 4 & -1 & 9 \\ 1 & 5 & -2 & 4 \\ 1 & 2 & -5 & 2 \end{pmatrix}$$

$$SSE = Y^T(I - H)T = Y^T(I - X(X^T X)^{-1} X^T)Y = 2.2742$$

$$\widehat{\sigma^2} = \frac{SSE}{df} = \frac{2.2742}{8-4} = 0.5685$$

$$\widehat{Var}(\widehat{\beta} \mid X) = \widehat{\sigma^2}(X^T X)^{-1} = \begin{pmatrix} 6.2637 & -1.0494 & 0.5512 & -0.1537 \\ -1.0494 & 0.1842 & -0.0990 & 0.0204 \\ 0.5512 & -0.0990 & 0.0565 & -0.0101 \\ -0.1537 & 0.0204 & -0.0101 & 0.0095 \end{pmatrix}$$

$$se(\widehat{\beta}_2) = \sqrt{\widehat{Var}(\widehat{\beta}_2 \mid X)} = \sqrt{0.0565} = 0.2376$$

$$\widehat{\beta} = (X^T X)^{-1} X^T Y = \begin{pmatrix} -3.1673 \\ 2.6774 \\ -0.3620 \\ 3.1373 \end{pmatrix}$$

$$t_{(0.025,4)} = 2.7764$$

$$obs.T.S. = \frac{\widehat{\beta}_2}{se(\widehat{\beta}_2)} = \frac{-0.3620}{0.2376} = -1.5233$$

$$|obs.T.S.| < 2.7764$$

Therefore, H_0 is not rejected at 5% significance level.

(iii)

$$H_0 : E(Y \mid X) = \beta_0 + \beta_1 (X_1 + X_2) + \beta_3 X_3$$

$$H_1 : E(Y \mid X) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$

Rewrite the full model as:

$$E(Y \mid X) = \beta_0 + \beta_1 (X_1 + X_2) + \beta_2 X_2 + \beta_3 X_3$$

To test $\beta_1 = \beta_2$,

Under H_1, $X = \begin{pmatrix} 1 & X_1 & X_2 & X_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 10 & 2 \\ 1 & 1 & 7 & 6 \\ 1 & 3 & 6 & 0 \\ 1 & 3 & 3 & 9 \\ 1 & 5 & 2 & 7 \\ 1 & 5 & -1 & 9 \\ 1 & 7 & -2 & 4 \\ 1 & 7 & -5 & 2 \end{pmatrix}$

$$SSE_{H_1} = Y^T (I - H) T = Y^T (I - X(X^T X)^{-1} X^T) Y = 2.2742$$

$$df_{H_1} = n - (p+1) = 8 - 4 = 4$$

Under H_0, $X = \begin{pmatrix} 1 & X_1 + X_2 & X_3 \end{pmatrix} = \begin{pmatrix} 1 & 11 & 2 \\ 1 & 8 & 6 \\ 1 & 9 & 0 \\ 1 & 6 & 9 \\ 1 & 7 & 7 \\ 1 & 4 & 9 \\ 1 & 5 & 4 \\ 1 & 2 & 2 \end{pmatrix}$

$$SSE_{H_0} = Y^T (I - H) T = Y^T (I - X(X^T X)^{-1} X^T) Y = 3.5935$$

$$df_{H_0} = n - (p+1) = 8 - 3 = 5$$

$$F_{(0.05,1,4)} = 7.7086$$

$$Obs.T.S. = \frac{\dfrac{SSE_{H_1} - SSE_{H_0}}{df_{H_0} - df_{H_1}}}{\dfrac{SSE_{H_1}}{df_{H_1}}} = 2.3206 < 7.7086$$

Therefore, H_0 is not rejected at 5% significance level.

Thus, $F = t^2$.