STAT3008 Exercise 10 Solutions

(2011-2012 2nd Semester)

Q1.

(i)

$$\left(X^T X\right)^{-1} = \frac{1}{266}\begin{pmatrix} 327 & -37 \\ -37 & 5 \end{pmatrix}$$

$$H = X\left(X^T X\right)^{-1} X^T = \frac{1}{266}\begin{pmatrix} 199 & 64 & 64 & 10 & -71 \\ 64 & 54 & 54 & 50 & 44 \\ 64 & 54 & 54 & 50 & 44 \\ 10 & 50 & 50 & 66 & 90 \\ -71 & 44 & 44 & 90 & 159 \end{pmatrix}$$

$$\hat{e} = \left(I_5 - H\right)Y = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 5 \\ 7 \end{pmatrix} - HY = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 5 \\ 7 \end{pmatrix} - \frac{1}{266}\begin{pmatrix} 136 \\ 946 \\ 946 \\ 1270 \\ 1756 \end{pmatrix} = \begin{pmatrix} 0.4887 \\ -1.5564 \\ 0.4436 \\ 0.2256 \\ 0.3985 \end{pmatrix}$$

$$\hat{e} \bullet 1 = 0.4887 - 1.5564 + 0.4436 + 0.2256 + 0.3985 = 0$$

(ii)

$$Var(e) = \sigma^2 I_5$$

(iii)

$$Var(\hat{e}) = Var\left[\left(I - H\right)Y\right] = \sigma^2\left(I - H\right)$$

$$= \sigma^2\begin{pmatrix} 0.2519 & -0.2406 & -0.2406 & -0.0376 & -0.2670 \\ -0.2406 & 0.7970 & -0.2030 & -0.1880 & -0.1654 \\ -0.2406 & -0.2030 & 0.7970 & -0.1880 & -0.1654 \\ -0.0376 & -0.1880 & -0.1880 & 0.7519 & -0.3383 \\ -0.2670 & -0.1654 & -0.1654 & -0.3383 & 0.4023 \end{pmatrix}$$

(iv)

$$\widehat{Var}(\hat{e}) = \widehat{\sigma^2}\left(I - H\right)$$

$$\widehat{\sigma^2} = \frac{\hat{e}^T \hat{e}}{n-(p+1)} = \frac{3.0677}{5-\left(1+1\right)} = 1.0226$$

$$\widehat{Var}(\hat{e}) = \begin{pmatrix} 0.2576 & -0.2460 & -0.2460 & -0.0384 & 0.2729 \\ -0.2460 & 0.8150 & -0.2076 & -0.1922 & -0.1691 \\ -0.2460 & -0.2076 & 0.8150 & -0.1922 & -0.1691 \\ -0.0384 & -0.1922 & -0.1922 & 0.7688 & -0.3460 \\ 0.2729 & -0.1691 & -0.1691 & -0.3460 & 0.4113 \end{pmatrix}$$

(v)

$$\widehat{Var}(e) = \widehat{\sigma^2} I_5 = \frac{\hat{e}^T \hat{e}}{n-(p+1)} I_5 = \frac{3.0677}{5-\left(1+1\right)} I_5 = 1.0226 I_5$$

Q2.

(i)

$$Cov\left(e, Y\right) = Cov\left(e, X\beta + e\right)$$
$$= Cov\left(e, e\right) = Var(e) = \sigma^2 I$$

(ii)

$$Cov\left(e, \hat{Y}\right) = Cov\left(e, X\left(X^T X\right)^{-1} X^T Y\right)$$

$$= Cov\left(e, X\left(X^T X\right)^{-1} X^T\left(X\beta + e\right)\right)$$

$$= Cov\left(e, X\beta + X\left(X^T X\right)^{-1} X^T e\right)$$

$$= Cov\left(e, X\left(X^T X\right)^{-1} X^T e\right)$$

$$= X\left(X^T X\right)^{-1} X^T Cov\left(e, e\right) = HVar(e) = \sigma^2 H$$

(iii)

$$Cov\left(\hat{e}, \hat{Y}\right) = Cov\left((I-H)Y, HY\right)$$
$$= (I-H)Cov\left(Y, Y\right)H^T$$
$$= (I-H)Var(Y)H$$
$$= (I-H)\sigma^2 IH$$
$$= \sigma^2(H - H^2)$$
$$= \sigma^2\left(H - H\right) = 0$$

(iv)

$$E\left[\sum_{i=1}^{n}\left(\hat{Y}_i-\overline{Y}\right)^2\right]$$

$$=E\left[Y^T\left(H-\frac{J}{n}\right)Y\right]$$

$$=E\left[tr\left\{Y^T\left(H-\frac{J}{n}\right)Y\right\}\right]$$

$$=E\left[tr\left\{\left(H-\frac{J}{n}\right)YY^T\right\}\right]$$

$$=tr\left\{\left(H-\frac{J}{n}\right)E\left(YY^T\right)\right\}$$

$$=tr\left\{\left(H-\frac{J}{n}\right)\left(\sigma^2 I+X\beta\beta^T X^T\right)\right\}$$

$$=\sigma^2 tr\left(H-\frac{J}{n}\right)+tr\left\{\left(H-\frac{J}{n}\right)X\beta\beta^T X^T\right\}$$

$$=\sigma^2(p+1-1)+tr\left(HX\beta\beta^T X^T\right)-\frac{1}{n}tr\left(JX\beta\beta^T X^T\right)$$

$$=p\sigma^2+tr\left(X\beta\beta^T X^T\right)-\frac{1}{n}tr\left(11^T X\beta\beta^T X^T\right)$$

$$=p\sigma^2+tr\left(\beta^T X^T X\beta\right)-\frac{1}{n}tr\left(\beta^T X^T 11^T X\beta\right)$$

$$=p\sigma^2+\beta^T X^T X\beta-\frac{1}{n}\left(1^T X\beta\right)^2$$