STAT3008 Applied Regression Anylysis

Suggested Midterm Solution

October 2013, by LI Chun

1.

ANOVA Table:

Source	Sum of Squuares	d.f.	Mean Square	F-statistics
Regression	74.178	1	74.178	3.3381
Residuals	2177.736	98	22.2218	
Total	2251.914	99		

Coefficient Table:

Variable	Coefficient	s.e.	t-statistics	p-value
Constant	0.5854	0.4729	1.238	0.2188
X	0.9002	0.4927	1.8271	0.0707
n=100	$R^2 = 0.03294$	$\hat{\sigma} = 4.714$		

Steps:

$$t_{\beta_0} = \frac{\hat{\beta}_0}{se(\hat{\beta}_0)} \Rightarrow se(\hat{\beta}_0) = \frac{\hat{\beta}_0}{t_{\beta_0}} = \frac{0.5854}{1.238} = 0.4729$$

$$se(\beta_1) = \frac{\hat{\sigma}}{\sqrt{SXX}} \Rightarrow SXX = \frac{\hat{\sigma}^2}{se(\beta_1)^2} = \left(\frac{4.714}{0.4927}\right)^2 = 91.5407$$

$$\hat{\sigma}^2 = \frac{RSS}{n-2} \Rightarrow RSS = (n-2) \times \hat{\sigma}^2 = 98 \times (4.714)^2 = 2177.736$$

$$\hat{\sigma}^2 = (4.714)^2 = 22.2218$$

$$R^2 = 1 - \frac{RSS}{SYY} \Rightarrow SYY = \frac{RSS}{1-R^2} = \frac{2177.7}{1-0.03294} = 2251.914$$

$$SSreg = SYY - RSS = 2251.914 - 2177.736 = 74.178$$

$$F = \frac{MSreg}{\hat{\sigma}^2} = \frac{74.178}{22.2218} = 3.3381$$

$$R^2 = \frac{(SXY)^2}{SXX \cdot SYY}$$

$$\Rightarrow \quad (SXY)^2 = R^2 \cdot SXX \cdot SYY = 0.03294 \times 91.5407 \times 2251.914 = 6790.31$$

$$\Rightarrow \quad SXY = 82.40302$$

$$\hat{\beta}_1 = \frac{SXY}{SXX} = \frac{84.40302}{91.5407} = 0.9002$$

$$t_{\beta_1} = \frac{\hat{\beta}_1}{se(\hat{\beta}_1)} = \frac{0.9002}{0.4927} = 1.8271$$

$$2 * (1 - pt(1.8271, 98)) = 0.0707$$

2.

i)

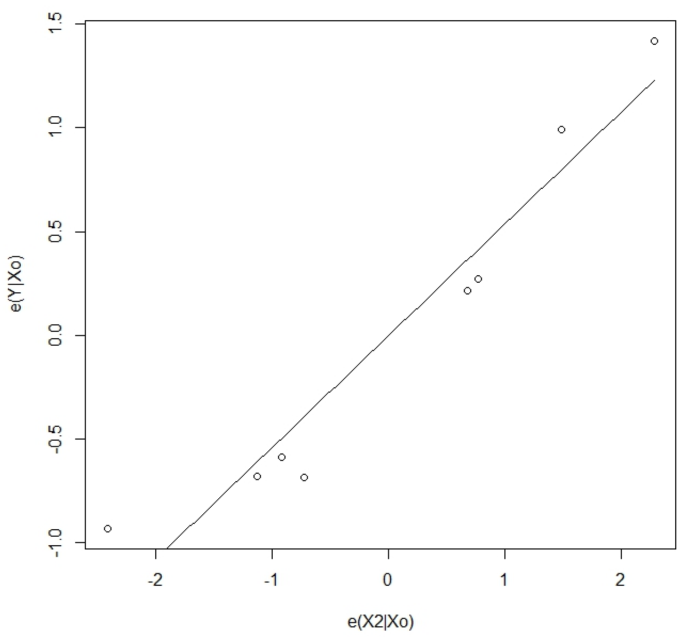
Name of the graph: Added-variable plot (for X_1)

y axis: Residual of the regression of y on all predictors except X_1

x axis: Residual of the regression of X_1 on all other predictors

ii)

The scatter plot:



The straight line in the plot is the OLS line. From the scatter plot we can conclude that a linear regression is not appropriate. We should look for a term/transformation that better fit the data, e.g., X_2^2.

iii)

$$\hat{\eta_0} = \frac{1}{8}\mathbf{1}'\hat{e}_{Y|X_1} - \hat{\eta_1}\left(\frac{1}{8}\mathbf{1}'\hat{e}_{X_2^2|X_1}\right) = 0$$

$$\hat{\eta_1} = \frac{SE_{31}E_1}{SE_{31}E_{31}} = \frac{51.19}{505.99} = 0.1012$$

$$\hat{\sigma}_1^2 = \frac{SE_1E_1 - \frac{(SE_1E_{31})^2}{SE_{31}E_{31}}}{n-2} = \frac{5.254 - \frac{51.19^2}{505.99}}{8-2} = 0.0752/6 = 0.01253$$

$$se(\hat{\eta_0}) = \hat{\sigma}_1\left[\frac{1}{8} + \frac{\left(\frac{1}{8}\mathbf{1}'\hat{e}_{X_2^2|X_1}\right)^2}{SE_{31}E_{31}}\right]^{\frac{1}{2}} = \sqrt{\frac{1}{8}\times 0.01253} = 0.03958$$

$$se(\hat{\eta_1}) = \sqrt{\hat{\sigma}_1^2\frac{1}{SE_{31}E_{31}}} = \sqrt{\frac{0.01253}{505.99}} = 0.004976$$

iv)

$$X'Y = \begin{pmatrix} \bar{Y} \\ SYX_1 \\ SYX_3 \end{pmatrix} = \begin{pmatrix} 20.125 \\ 2605.2 \\ 255.98 \end{pmatrix}$$

$$\begin{pmatrix} \hat{\gamma}_0 \\ \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{pmatrix} = (X'X)^{-1}X'Y = \begin{pmatrix} 0.4592 \\ 0.0079 \\ 0.1009 \end{pmatrix}$$

$$\begin{aligned}\hat{\sigma}_2^2 &= \frac{Y'[I - X(X'X)^{-1}X']Y}{n-3} \\ &= \frac{SYY - Y'X(X'X)^{-1}X'Y}{n-3} \\ &= \frac{56.14 - 55.75}{8-3} \\ &= 0.078\end{aligned}$$

$$var(\begin{pmatrix} \hat{\gamma}_0 \\ \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{pmatrix}) = \hat{\sigma}_2^2(X'X)^{-1} \Rightarrow se(\hat{\gamma}_2) = \sqrt{0.078 \times 0.001976} = 0.01241$$

v)

No, they differ in their degree of freedom.

$$se(\hat{\eta_1}) = \sqrt{\hat{\sigma}_1^2\frac{1}{SE_{31}E_{31}}},\ \hat{\sigma}_1^2 = \frac{Y'[I - X(X'X)^{-1}X']Y}{n-2}$$

$$se(\hat{\gamma}_2) = \sqrt{\hat{\sigma}_2^2 \frac{1}{SE_{31}E_{31}}},\ \hat{\sigma}_2^2 = \frac{Y'[I - X(X'X)^{-1}X']Y}{n-3}$$

vi)

Let $x_\star = [1,\ 200,\ 0]'$, then

$$\hat{y}_\star = (\hat{\gamma}_0, \hat{\gamma}_1, \hat{\gamma}_2)\, x_\star = 2.0392$$

A 90% prediction interval for Bosco's GPA is

$$\begin{aligned}
&\hat{y}_\star \pm t(0.05, 5)\hat{\sigma}_2\sqrt{1 + x_\star'(X'X)^{-1}x_\star} \\
&= 2.0392 \pm 2.0150 \times \sqrt{0.078}\sqrt{1 + 1.053} \\
&= [1.233, 2.846]
\end{aligned}$$

3.

(a)

$$\begin{aligned}
E(\sum_{i-1}^{n} \hat{y_i}^2) &= E(\hat{Y}'\hat{Y}) = E(Y'X(X'X)^{-1}X'X(X'X)^{-1}X'Y) \\
&= E(Y'X(X'X)^{-1}X'Y) \\
&= tr(X(X'X)^{-1}X'X\beta\beta'X') + \sigma^2 tr(X(X'X)^{-1}X') \\
&= \beta'X'X\beta + p\sigma^2
\end{aligned}$$

(b)

$$\begin{aligned}
E(\sum_{i-1}^{n}(\hat{y}_i - \bar{y})(y_i - \hat{y}_i)) &= E((\hat{Y} - \bar{y}\mathbf{1})'(Y - \hat{Y})) \\
&= E((Y'H - \frac{1}{n}Y'\mathbf{1}\mathbf{1}')(Y - HY)) \qquad\qquad H = X(X'X)^{-1}X' \\
&= E(Y'(H - \frac{1}{n}\mathbf{1}\mathbf{1}')(I - H)Y) \\
&= E(Y'[H(I - H) - \frac{1}{n}\mathbf{1}\mathbf{1}'(I - H)]Y) \\
&= 0 \qquad\qquad H(I - H) = 0,\ \frac{1}{n}\mathbf{1}\mathbf{1}'(I - H) = 0
\end{aligned}$$

4.

$$
\begin{aligned}
F\text{-statistics} &= \frac{SSreg/(p+2)}{RSS_{AH}/(n-p-3)} \\
&= \frac{n-p-3}{p+2}\frac{SSreg}{SYY-SSreg} \\
&= \frac{n-p-3}{p+2}\frac{1}{(SYY-SSreg)/SSreg} \\
&= \frac{n-p-3}{p+2}\frac{1}{\frac{SYY}{SSreg}-1} \\
&= \frac{n-p-3}{p+2}\frac{1}{\frac{1}{R^2}-1}
\end{aligned}
$$

5.

Method 1.

Let $\hat{\beta} = (X'X)^{-1}X'Y$. For each β,

$$
\begin{aligned}
RSS(\beta) &= (Y - X\beta)'(Y - X\beta) \\
&= (Y - X\hat{\beta} + X\hat{\beta} - X\beta)'(Y - X\hat{\beta} + X\hat{\beta} - X\beta) \\
&= (Y - X\hat{\beta})'(Y - X\hat{\beta}) + (\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta) + 2(\hat{\beta} - \beta)'X'(Y - X\hat{\beta}) \\
&= (Y - X\hat{\beta})'(Y - X\hat{\beta}) + (\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta) + 2(\hat{\beta} - \beta)'X'(I - H)Y \\
&= (Y - X\hat{\beta})'(Y - X\hat{\beta}) + (\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta). \qquad X'(I-H) = 0
\end{aligned}
$$

We have $RSS(\beta) \geq RSS(\hat{\beta})$, and the equality holds if and only if

$$
\begin{aligned}
& (\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta) = 0 \\
\Leftrightarrow\ & X(\hat{\beta} - \beta) = 0 \\
\Leftrightarrow\ & X'X\beta = X'X\hat{\beta} = X'Y \\
\Leftrightarrow\ & \beta = \hat{\beta} = (X'X)^{-1}X'Y
\end{aligned}
$$

Therefore, $\hat{\beta}$ is the unique minimizer.

Method 2.

Let

$$\frac{\partial RSS(\beta)}{\partial \beta} = -2X'Y + 2X'X\beta = 0.$$

When $X'X$ is invertible, the only solution to the above equation is $\hat{\beta} = (X'X)^{-1}X'Y$. And we have

$$\frac{\partial^2 RSS(\beta)}{\partial \beta^2} = 2X'X.$$

As $X'X$ is positive definite, $\hat{\beta}$ is the minimizer.